partial identification in linear models

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a multivariat normal i.i.d. error ε and a dependend n dimensional random variable $Y^* = (Y_1^*, \dots, Y_n^*)$, that is only known to lie in the intervall $[\underline{\mathrm{Y}}, \overline{\mathrm{Y}}]$ of the known random variables $\underline{\mathrm{Y}}$ and $\overline{\mathrm{Y}}$.

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which stands for a possible sample of Y compatible with the interval-valued observed data.

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to get an estimate $\hat{\beta}(y)$ for all y.

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\end{aligned}
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The calculation of $\hat{\beta}(y)$ for all $y \in [\underline{y}, \overline{y}]$ is nothing else than the computation of the linear image of the 2 dimensional minkowski mean of the n line segments p_i formed by the points $(\underline{y}_i, x_i \cdot \underline{y}_i)$ and $(\overline{y}_i, x_i \cdot \overline{y}_i)$ under the mapping induced by the matrix P:

$$\hat{S} = P \cdot \left(\frac{1}{n} \bigoplus_{i=1}^{n} p_i \right)$$

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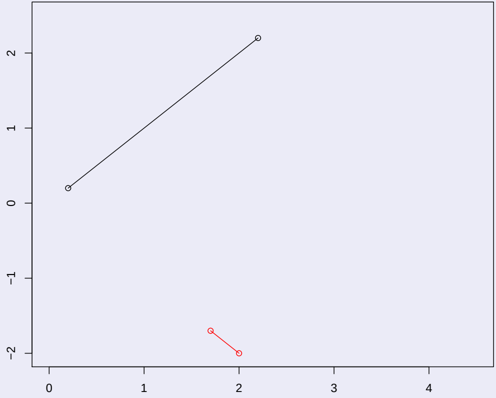
The Minkowski Mean of n *pointsets* $A_i, \ldots, A_n$ *is defined as:*

$$\frac{1}{n} \bigoplus_{i=1}^{n} A_i := \left\{ \frac{1}{n} \sum_{i=1}^{n} a_i \middle| a_i \in A_i, i = 1, \ldots, n \right\}$$

Example

The Minkowski Sum of two line segments in $\mathbb{R}^2$:



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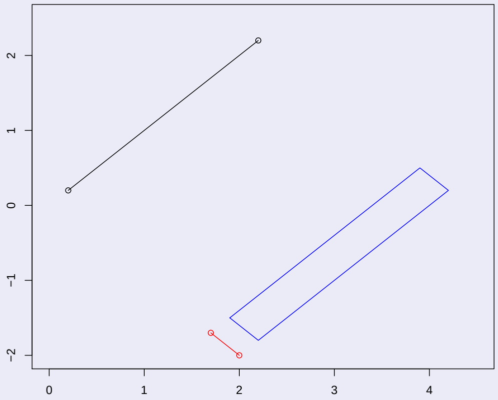

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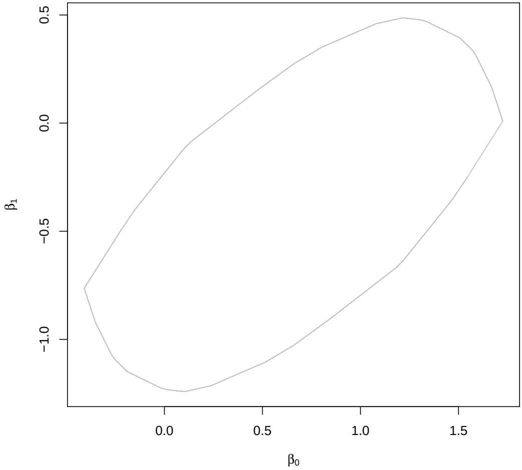

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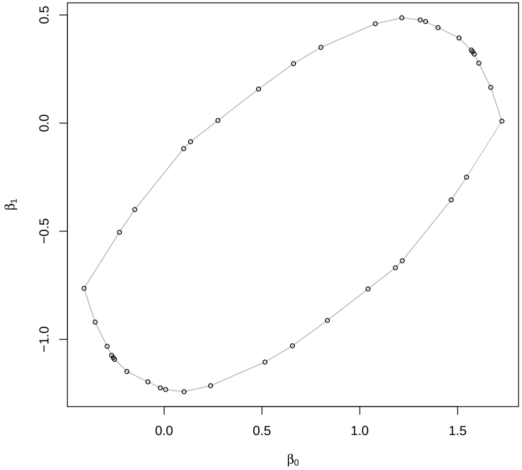
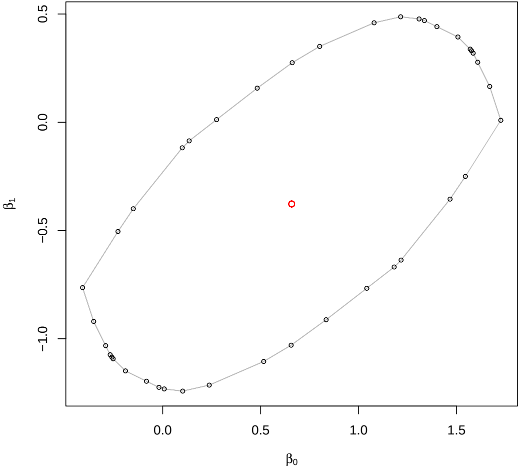
a) it has finite many extremepoints.



The set-valued estimate $\hat{S}$ has the following properties:

b) it is central symmetric with the center $\hat{\beta}(\frac{\underline{y}+\overline{y}}{2})$.



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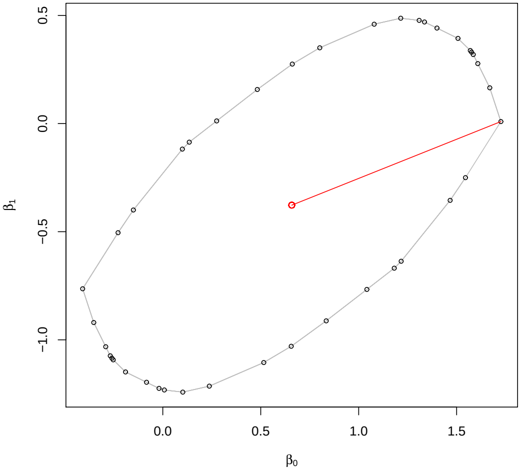

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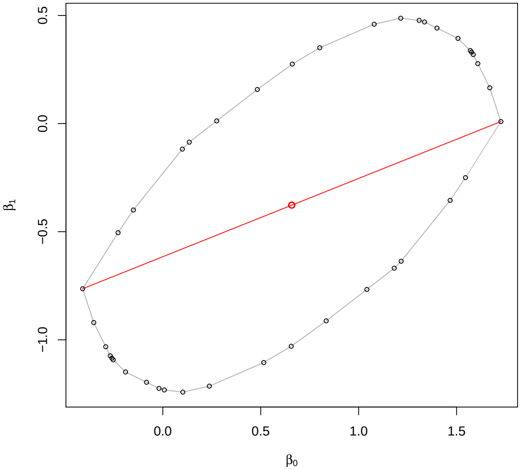

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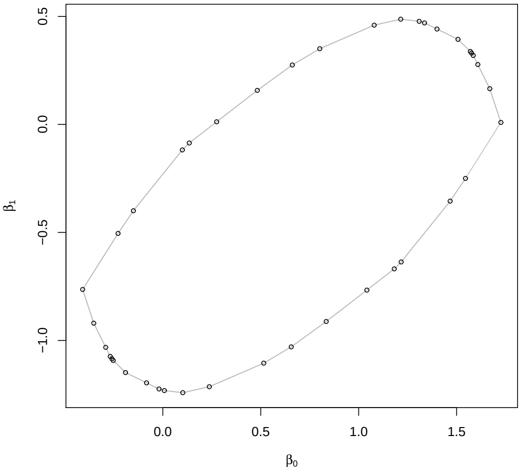

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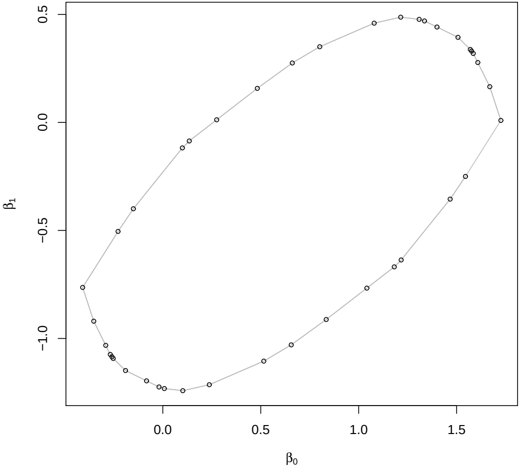
c) in geometry it is, as the Minkowski Sum of n line segments, called a zonotope.



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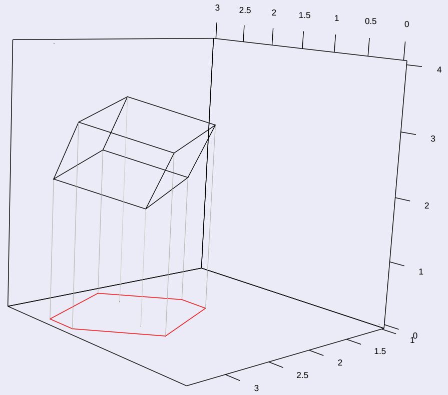

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$$\hat{S} = \mathrm{co}\,\{A \cdot y | y \text{ is a pseudodata } \}$$

We have two perspectives on $\hat{S}$

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we could understand $\hat{S}$ as a point-estimator, which estimates the linear image of the (set-valued) so called Aumann Expectation
$\left\{ \begin{pmatrix} \mathbb{E}(Y) \\ \mathbb{E}(X \cdot Y) \end{pmatrix} \middle| \, y \in [\underline{y}, \overline{y}] \right\}$ under P.

So $\hat{S}$ could at first hand be seen as a (set-valued) pointestimator for a (set-valued) parameter (the Aumann Expectation). Here we can use random set theory to analyze the estimator.

(2) $\hat{S}$ as the collection of all precise pointestimators obtained by all possible data-completions $y \in [\underline{\mathrm{y}}, \overline{\mathrm{y}}]$.

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and a metric d in $\mathbb{R}^d$ (e.g. the euclidean metric).

This approach is develepoed in Beresteanu, Molinari 2008:

There the authors estimate $\hat{S}$ and draw bootstrap-samples from the data to estimate further $\hat{S}^\star$ and look on the distribution of $H(\hat{S}, \hat{S}^\star)$. From this distribution they obtain critical value c_α and construct the confidence collection

$$HCR = \bigcup_{\substack{S \subset \mathbb{R}^d \\ dH(S,\hat{S}) \leq c_\alpha}} S.$$

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If one is in the situation, that there is a precise parameter β behind the scenes, it would be sufficient, that a confidenceregion covers not necessarily the whole sharp identification region but only the true parameter β with at least probability α, which is a weaker demand. So in this situation HCR is a (conservative) confidenceregion for the true parameter β.

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$$CE(y) := \left\{ \beta \mid (\beta - \hat{\beta}(y))'(X'X)(\beta - \hat{\beta}(y)) \leq (p+1) \cdot \hat{\sigma}^2(y) \cdot F_{1-\alpha}(p+1, n-p+1) \right\}.$$

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Let a partially identified linear model $y = \beta_0 + \beta_1 \cdot x$ *be given.*
Under some not too strong conditions the simple confidenceregion SCR is a subset of the ellipsoid-type-confidenceregion

$$ECR := \mathrm{co}\left(\bigcup_{c \in \{x_1, \ldots, x_n\}} CE(y^u_{\geq c}) \cup CE(y^l_{\geq c}) \right)$$

Definition

Let the functions f, g and esd be defined as:

$$
\begin{aligned}
f: &\quad \hat{S} \longrightarrow [\underline{y}, \overline{y}] : \beta \mapsto Q^{-1}(\beta) = \{y \in [\underline{y}, \overline{y}] | Qy = \beta\} \\
g: &\quad \hat{S} \longrightarrow \mathbb{R} : \beta \mapsto \sup_{y \in f(\beta)} esd(y)
\end{aligned}
$$

$$
\begin{aligned}
esd : [\underline{y}, \overline{y}] \longrightarrow \mathbb{R} : y \mapsto esd(y) &:= sd(y - X\hat{\beta}(y)) \\
&= sd(\varepsilon) \\
&= \frac{n}{n-p-1}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\right)^2}.
\end{aligned}
$$

Lemma

Let a partially identified linear model $y = \beta_0 + \beta_1 \cdot x$ *with* $x_i, \underline{y}_i$ *and* $\overline{y}_i$ *i.i.d with existing expectations and variances and a nondegenerate sharp identification region* S *(meaning* S *has nonempty interior) be given.*
If the function g *is greater than a positive constant* c *(independent from* n*) with probability* 1 *for all* $\hat{\beta}$ *of the boundary* $\partial\hat{S}$ *of the Sir-estimator* $\hat{S}$*, then the simple confidenceregion is a subset of the ellipsoid-type-confidenceregion*

$$ECR := \text{co}\left(\bigcup_{c \in \{x_1, \ldots, x_n\}} CE(y^u_{\geq c}) \cup CE(y^l_{\geq c})\right)$$

with arbitrary high probability $p < 1$*, if* $n = n(p)$ *is large enough.*

short simulationstudy:

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for 4 coarsening-processes:

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- coarsening 1: $y = 10 \cdot x + 10 + \varepsilon, \quad \varepsilon \sim N(0,1)$
  $\underline{y} = y - \exp(\varepsilon_2), \quad \bar{y} = y + \exp(\varepsilon_2), \quad \varepsilon, \varepsilon_2 \sim N(0,1),$

short simulationstudy:

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- coarsening 3: $\underline{\mathrm{y}} = y - \varepsilon^2 \cdot 10^{-5}, \quad \bar{\mathrm{y}} = y + \varepsilon_2^2 \cdot 10^{-5} \cdot p, \quad p \sim B(n, 0.05)$

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- coarsening 4: $\underline{y} = y$

coarsening	N	SIR	HCR	ECR
1	10	0.96	1	1
1	100	1	1	1
1	1000	1	1	1
2	10	0.43	1	0.99
2	100	0.59	0.99	0.99
2	1000	0.80	1	1
3	10	0		1
3	100	0	0.92	0.95
3	1000	0	0.7?	
4	10	0.22	1	1
4	100	0.54		1
4	1000	0.82	1	1

coarsening	N	SIR	HCR	ECR
1	10	7.18	102.33	55.40
1	100	6.22	14.31	13.07
1	1000	6.14		8.08
2	10	5.33	25.81	22.90
2	100	5.60	8.79	8.67
2	1000	5.62	6.57	6.51
3	10	$7 \cdot 10^{-11}$		3.37
3	100	$6.29 \cdot 10^{-11}$	0.19	0.19
3	1000	$6.39 \cdot 10^{-11}$	0.02	0.02
4	10	9.90	15848.89	10485.69
4	100	1.22		87.30
4	1000	0.31	1.48	1.57

One „real-world-example“:

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- sample from East Germany ($n = 1077$)

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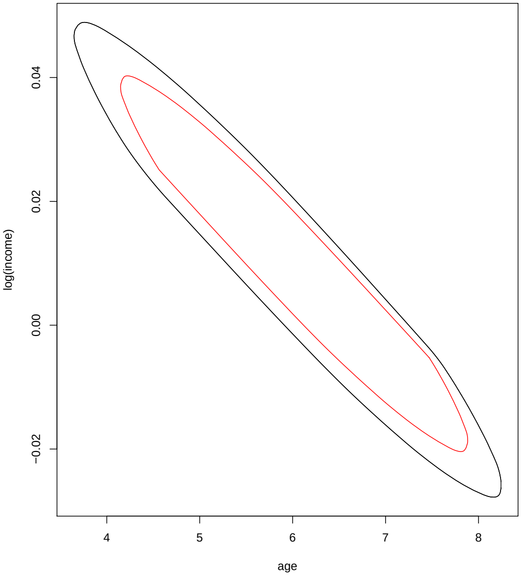
- sample from East Germany ($n = 1077$)
- age (x, precise) and logarithm of income (y, interval-valued)



0.04
0.02
0.00
−0.02
log(income)
4
5
6
7
8
age

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