On Sharp Identification Regions for Regression Under Interval Data

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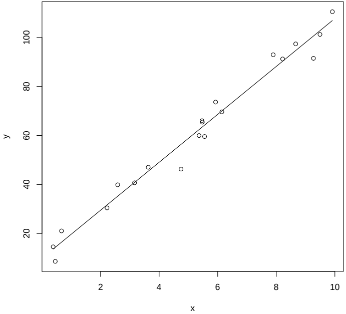
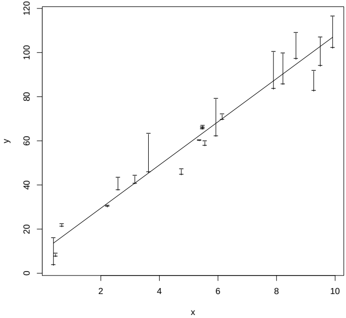
Situation

Situation: Standard simple linear model $Y = \beta_0 + \beta_1 X + \varepsilon$ with interval-censored outcomes.

- Y unobserved, only $\underline{Y}$ and $\overline{Y}$ observed and all we know is $Y \in [\underline{Y}, \overline{Y}]$.
- model partially identified, there are many parameters (β_0, β_1) leading to the same distribution of the observable variables.





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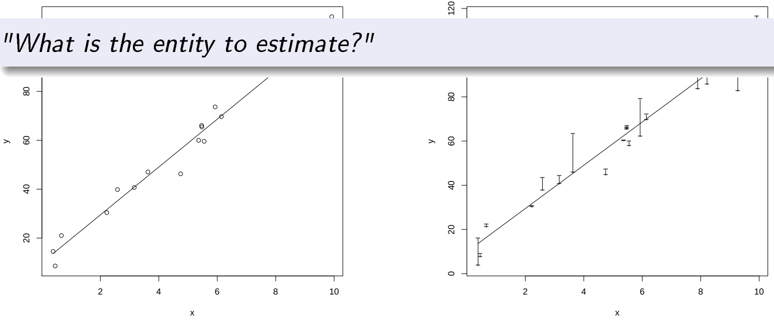

"What is the entity to estimate?"

Identification regions for the simple linear model

a) Sharp Marrow Region:

$$SMR(\underline{Y}, \overline{Y}) = \{\beta \mid \mathbb{E}(\underline{Y} \mid X) \leq \beta_0 + \beta_1 X \leq \mathbb{E}(\overline{Y} \mid X)\}$$

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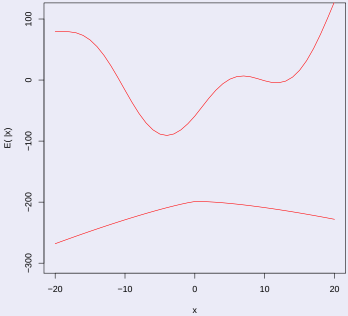

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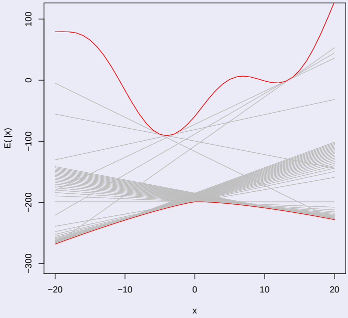

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b) Sharp Collection Region:

$$SCR(\underline{\mathrm{Y}}, \overline{\mathrm{Y}}) = \{\operatorname{argmin} \mathbb{E}((\beta_0 + \beta_1 X - Y)^2) \mid Y \in [\underline{\mathrm{Y}}, \overline{\mathrm{Y}}]\}$$

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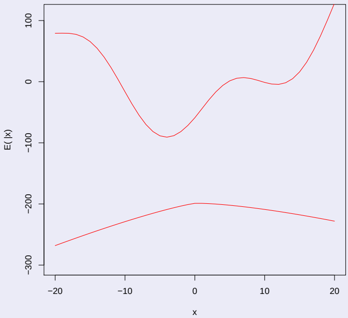

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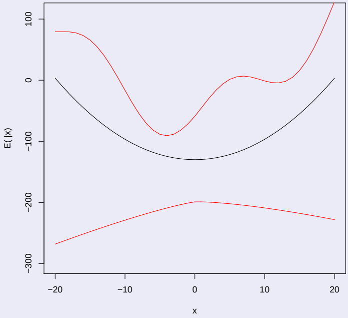

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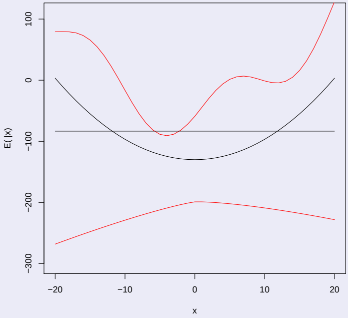

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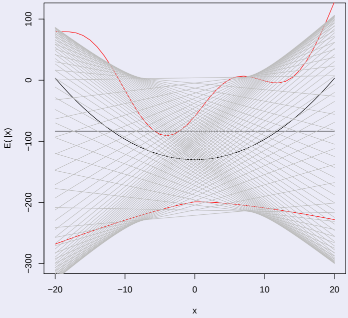

Sharp Setloss Region

c) Sharp Setloss Region:

$$L_S(\underline{Y}, \overline{Y}, \Gamma) = \int \left[\mathbb{E}(\overline{Y} \mid x) - \sup_{\beta \in \Gamma} (\beta_0 + \beta_1 x) \right]^2 + \left[\mathbb{E}(\underline{Y} \mid x) - \inf_{\beta \in \Gamma} (\beta_0 + \beta_1 x) \right]^2 d\mathbb{P}(x)$$

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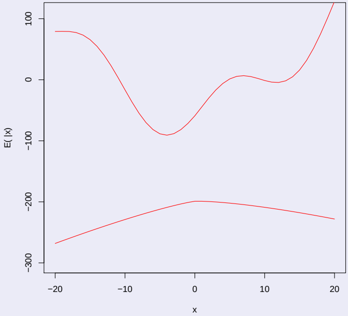

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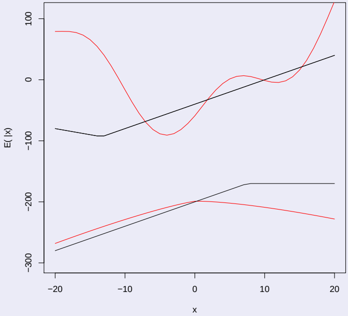

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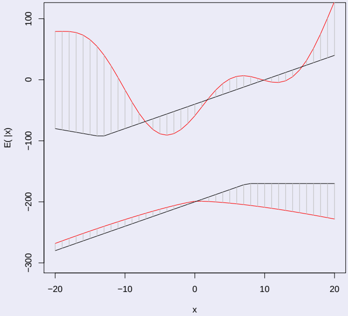

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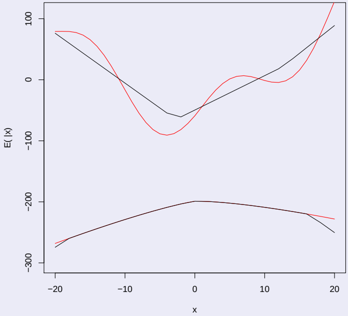

Relations between SMR, SCR and SSR

- $SSR \supseteq SMR \subseteq SCR$

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- $SSR \subseteq SCR$?

Relations between SMR, SCR and SSR

- $SSR \supseteq SMR \subseteq SCR$
- $SSR \subseteq SCR$? *"often approximately"*

region	well-specified case	misspecified case
SMR	• true parameters $(\beta_0, \beta_1) \in SMR$ • not "well-estimable"	• ?
SSR	• true parameters $(\beta_0, \beta_1) \in SSR$ • "well-estimable" (e.g. replace expectations by means)	• good description of the bounds $\underline{Y}, \overline{Y}$ and thus good description of Y
SCR	• true parameters $(\beta_0, \beta_1) \in SCR$ • "well-estimable" (e.g. replace expectations by means)	• set of all possible good linear descriptions for every $Y \in [\underline{Y}, \overline{Y}]$

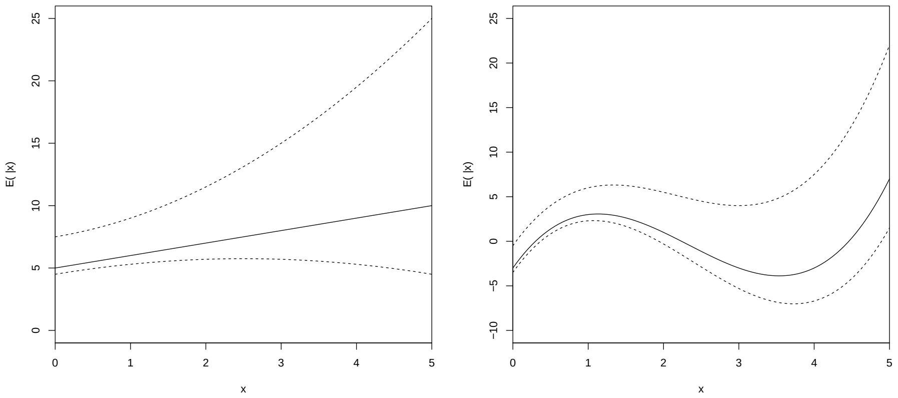

Figure: The conditional expectations $\mathbb{E}(Y \mid x)$ (black), $\mathbb{E}(\underline{Y} \mid x)$ (dashed) and $\mathbb{E}(\overline{Y} \mid x)$ (dashed) for a well-specified (left) and a misspecified (right) situation.

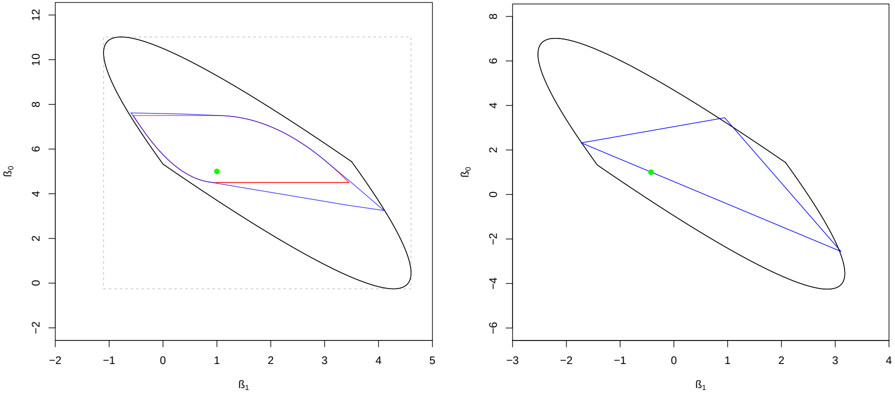

Figure: The identification regions SMR (red), SCR (black) and SSR (blue) for a well-specified (left) and a misspecified (right) situation. The true Parameter is dotted green (for the misspecified situation the green point is the best linear predictor for y). In the misspecified case SMR is empty.

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region	well-specified case	misspecified case
SMR	• truth $\in SMR$ • not "well-estimable" • monotone • idempotent	• ?
SSR	• truth $\in SSR$ • "well-estimable" • not monotone • idempotent	• good description of the bounds $\underline{Y}, \overline{Y}$ and thus good description of Y
SCR	• truth $\in SCR$ • "well-estimable" • monotone • not idempotent and thus maybe too rough	• set of all possible good linear descriptions for every $Y \in [\underline{Y}, \overline{Y}]$