Linear models and partial identification: Imprecise linear regression with interval data

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a multivariat normal i.i.d. error ε and a dependend n dimensional random variable $Y^* = (Y_1^*, \dots, Y_n^*)$, that is only known to lie in the intervall $[\underline{\mathrm{Y}}, \overline{\mathrm{Y}}]$ of the known random variables $\underline{\mathrm{Y}}$ and $\overline{\mathrm{Y}}$.

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which stands for a possible sample of Y compatible with the interval-valued observed data.

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to get an estimate $\hat{\beta}(y)$ for all y.

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$$=: P \cdot \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} y_i \\ \frac{1}{n}\sum_{i=1}^{n} x_i \cdot y_i \end{pmatrix}$$

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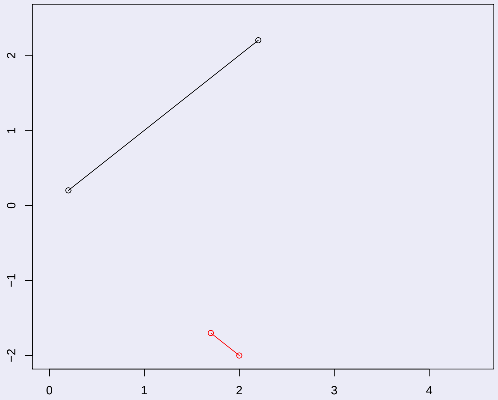
The Minkowski Mean of n *pointsets* $A_1, \ldots, A_n$ *is defined as:*

$$\frac{1}{n} \bigoplus_{i=1}^{n} A_i := \left\{ \frac{1}{n} \sum_{i=1}^{n} a_i \middle| a_i \in A_i, i = 1, \ldots, n \right\}$$

Example

The Minkowski Sum of two line segments in $\mathbb{R}^2$:



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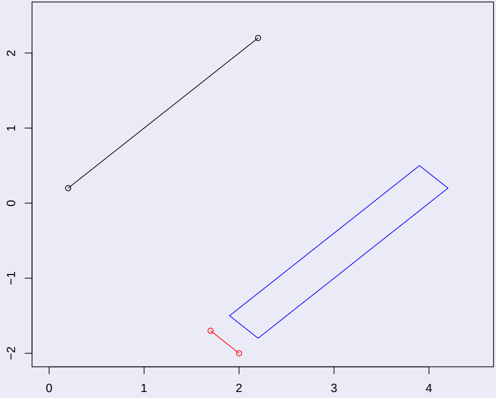

The calculation of $\hat{\beta}(y)$ for all $y \in [\underline{y}, \overline{y}]$ is nothing else than the computation of the linear image of the 2 dimensional minkowski mean of the n line segments p_i formed by the points $(\underline{y}_i, x_i \cdot \underline{y}_i)$ and $(\overline{y}_i, x_i \cdot \overline{y}_i)$ under the mapping induced by the matrix P:

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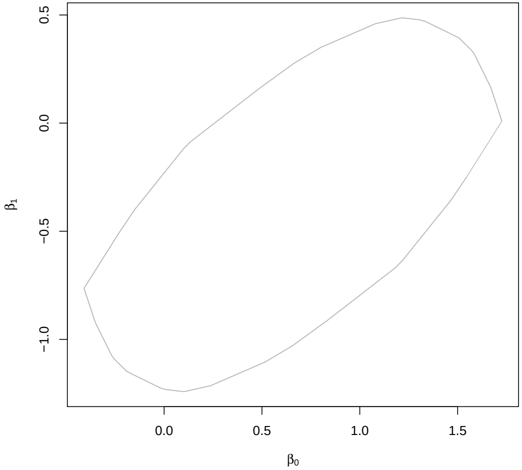

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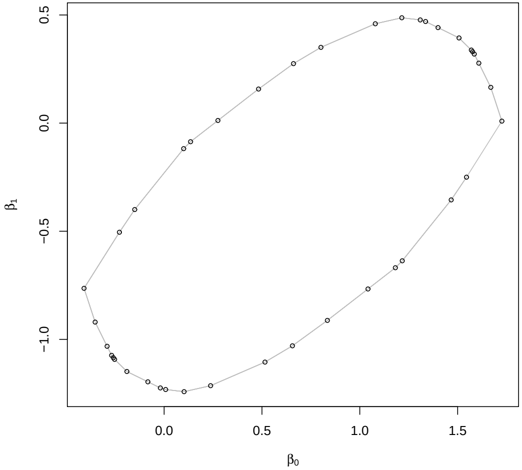

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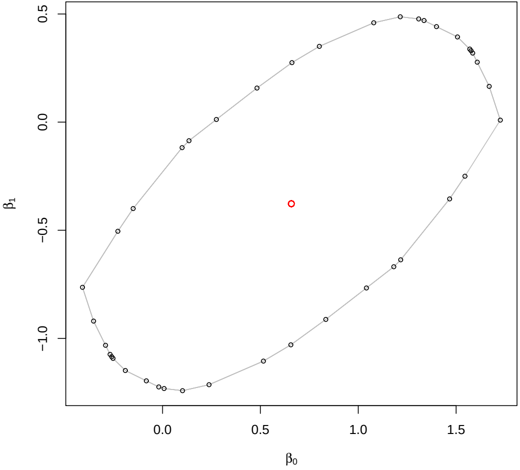
b) it is central symmetric with the center $\hat{\beta}(\frac{\underline{y}+\overline{y}}{2})$.



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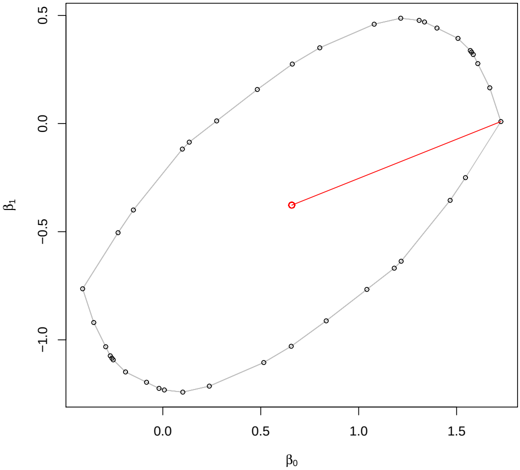

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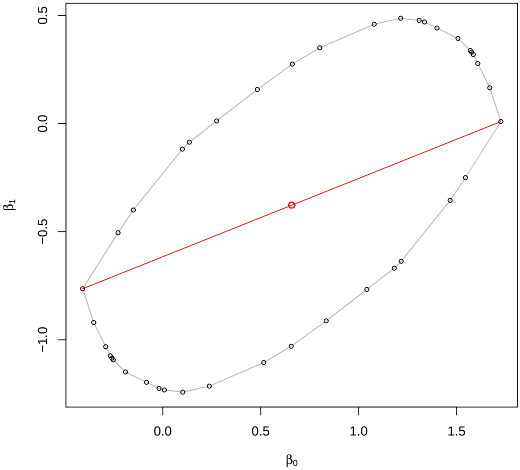

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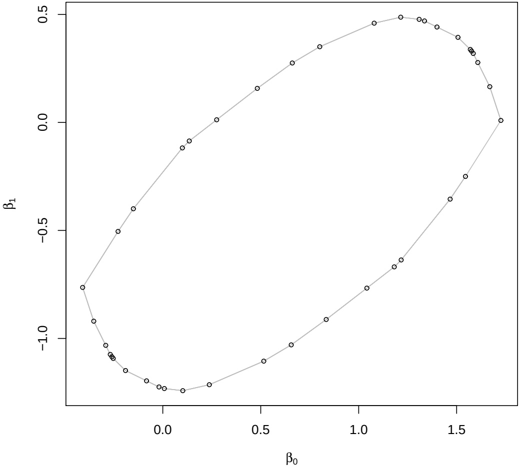
c) its facets are central symmetric, too.



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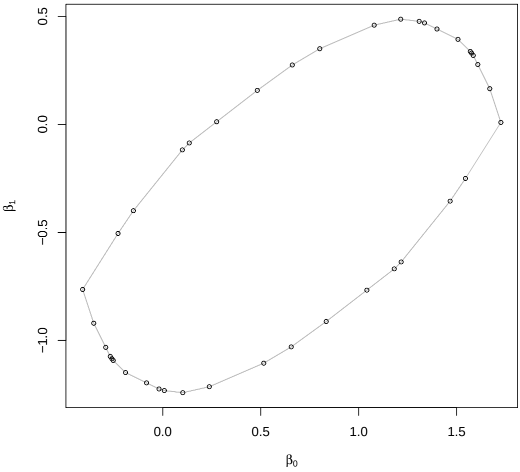
c) in geometry it is, as the Minkowski Sum of n line segments, called a zonotope.



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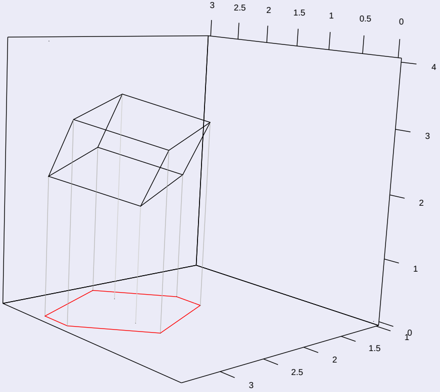

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$\left\{ \begin{pmatrix} \mathbb{E}(Y) \\ \mathbb{E}(X \cdot Y) \end{pmatrix} \middle| Y \in [\underline{Y}, \overline{Y}] \right\}$ under P (often called the sharp identification region).

So $\hat{S}$ could at first hand be seen as a (set-valued) pointestimator for a (set-valued) parameter (the Aumann Expectation under P). Here we can use random set theory to analyze the estimator.

(2) $\hat{S}$ as the collection of all precise pointestimators obtained by all possible data-completions $y \in [\underline{\mathrm{y}}, \overline{\mathrm{y}}]$.

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and a metric d in $\mathbb{R}^d$ (e.g. the euclidean metric).

This approach is develepoed in Beresteanu, Molinari 2008*:*

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This confidenceregion asymptotically covers the whole sharp identification region with probability at least $1 - \alpha$ (under some regularity assumptions).

There the authors estimate $\hat{S}$ and draw bootstrap-samples from the data to estimate further $\hat{S}^\star$ and look on the distribution of $dH(\hat{S}^\star, \hat{S})$. From this distribution they obtain a critical value c_α and construct the confidence collection

$$HCR = \bigcup_{\substack{S \subset \mathbb{R}^d \\ dH(S,\hat{S}) \leq c_\alpha}} S.$$

This confidenceregion asymptotically covers the whole sharp identification region with probability at least $1-\alpha$ (under some regularity assumptions).

If one is in the situation, that there is a precise parameter β behind the scenes, it would be sufficient, that a confidenceregion covers not necessarily the whole sharp identification region but only the true parameter β with at least probability $1-\alpha$, which is a weaker demand. So in this situation HCR is a (conservative) confidenceregion for the true parameter β.

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$$CE(y) := \left\{\beta \middle| (\beta - \hat{\beta}(y))'(X'X)(\beta - \hat{\beta}(y)) \leq (p+1) \cdot \hat{\sigma}^2(y) \cdot F_{1-\alpha}(p+1, n-p+1)\right\}.$$

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Lemma

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Lemma

Let a partially identified linear model $y = \beta_0 + \beta_1 \cdot x + \varepsilon$ be given.
Under some not too strong conditions the simple confidenceregion SCR is a subset of the ellipsoid-type-confidenceregion

$$ECR := \text{co}\left(\bigcup_{c \in \{x_1, \ldots, x_n\}} CE(y^u_{\geq c}) \cup CE(y^l_{\geq c}) \right)$$

with arbitrary high probability $p < 1$, if $n = n(p)$ is large enough.

One „real-world-example“:

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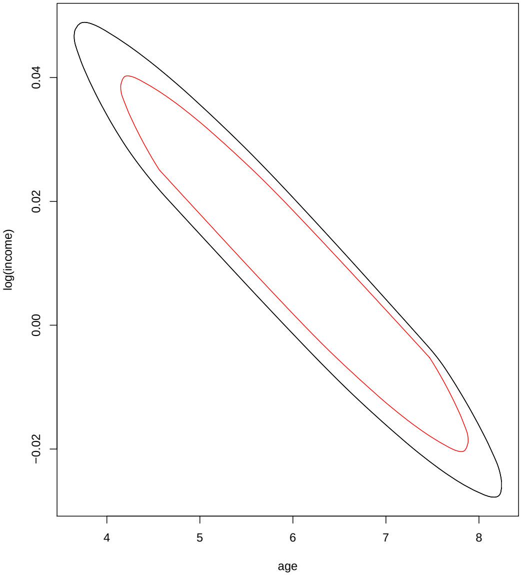
- age (x, precise) and logarithm of income (y, interval-valued)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log(income)
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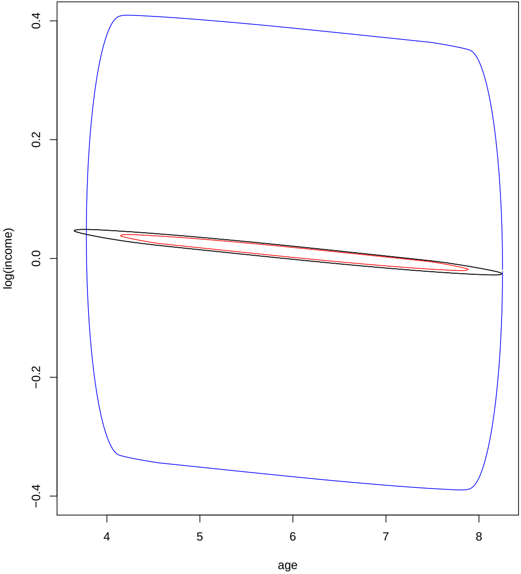
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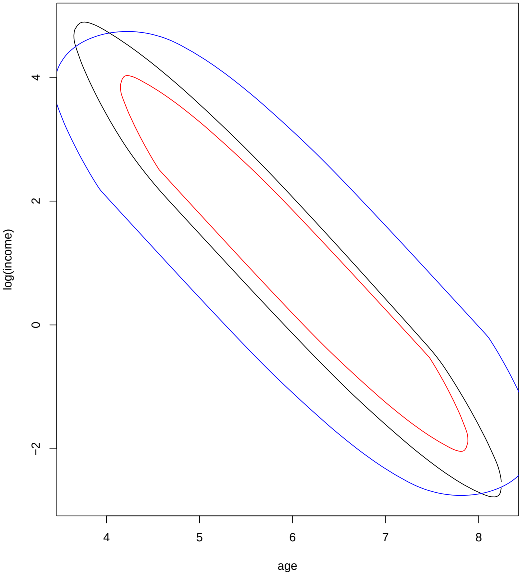
log(income)
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for 4 coarsening-processes:

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  $\underline{y} = y - \exp(\varepsilon_2), \quad \overline{y} = y + \exp(\varepsilon_3), \quad \varepsilon, \varepsilon_2, \varepsilon_3 : i.i.d., \sim N(0,1),$

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- coarsening 3: $\underline{y} = y - \varepsilon_2^2 \cdot 10^{-5}, \quad \bar{y} = y + \varepsilon_3^2 \cdot 10^{-5} \cdot p, \quad p \sim B(n, 0.05)$

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- coarsening 4: $\underline{y} = p \cdot y + (1-p) \cdot min\{-200, y\}$
  $\overline{y} = y + \varepsilon^2 \cdot q$
  $p \sim B(n, u_1), \quad u_1 \sim u[0,1]$
  $q \sim B(n, u_2^2), \quad u_2 \sim u[0,1]$

Covering Probabilities:

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coarsening	N	SIR	HCR	ECR
1	10	0.96	1	1
1	100	1	1	1
1	1000	1	1	1
2	10	0.43	1	0.99
2	100	0.59	0.99	0.99
2	1000	0.80	1	1
3	10	0	0.93	1
3	100	0	0.92	0.95
3	1000	0	0.96	0.95
4	10	0.22	1	1
4	100	0.54	1	1
4	1000	0.82	1	1

Areas:

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coarsening	N	SIR	HCR	ECR
1	10	7.18	102.33	55.40
1	100	6.22	14.31	13.07
1	1000	6.14	8.62	8.08
2	10	5.33	25.81	22.90
2	100	5.60	8.79	8.67
2	1000	5.62	6.57	6.51
3	10	$7 \cdot 10^{-11}$	3.97	3.37
3	100	$6.29 \cdot 10^{-11}$	0.19	0.19
3	1000	$6.39 \cdot 10^{-11}$	0.02	0.02
4	10	9.90	15848.89	10485.69
4	100	1.22	142.84	87.30
4	1000	0.31	1.48	1.25

An Idea of robustification

1) a bad idea:

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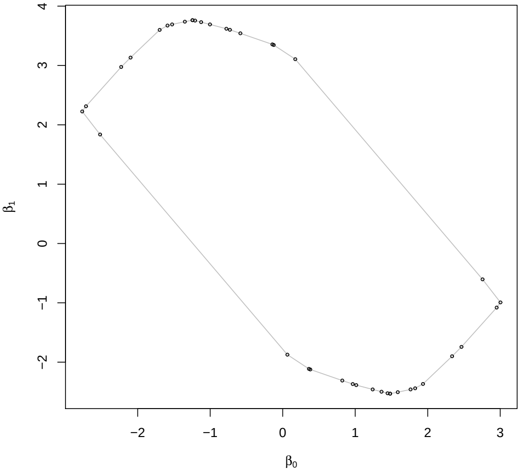
apply a robust method to all pseudodata.



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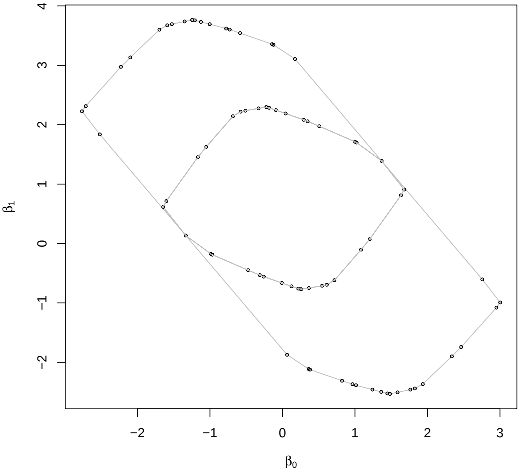

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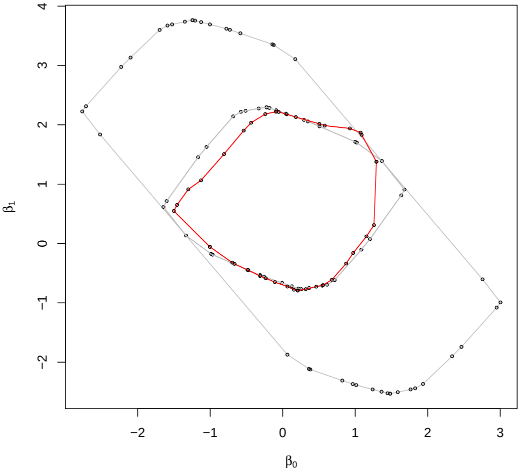

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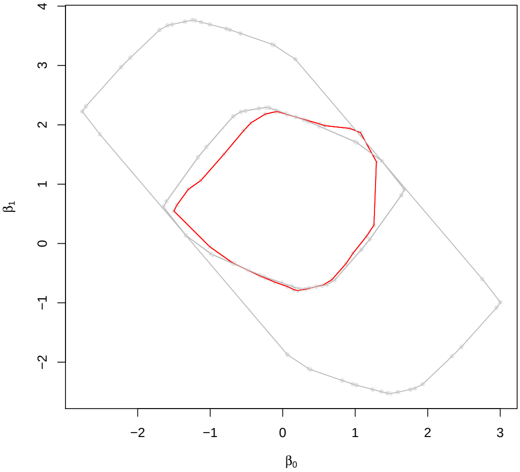

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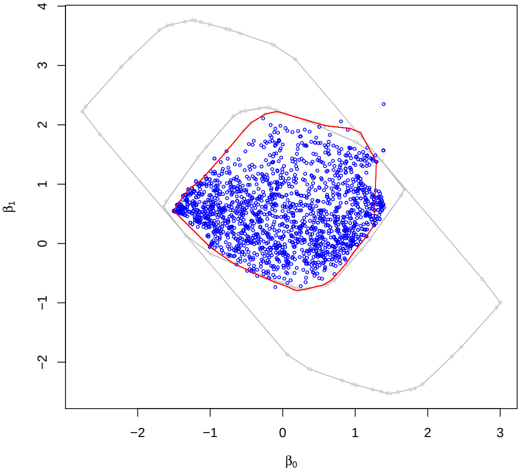

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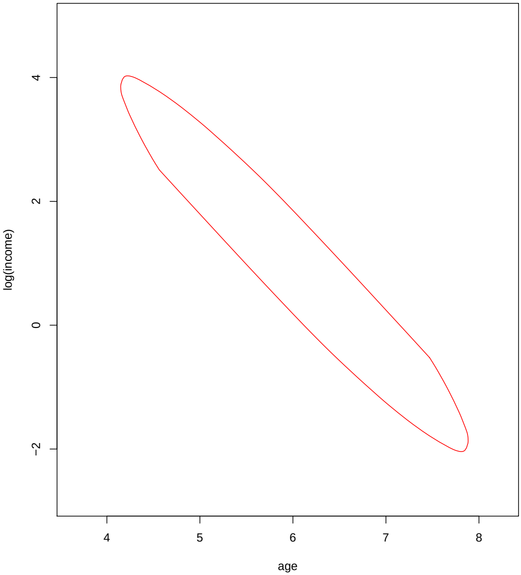
  All properties of the unweighted zonotope-estimator are kept.
- For confidenceregions use the Hausdorff-based approach of Beresteanu and Molinari, but maybe with another d in the definition of the Hausdorff-distance.


log(income)
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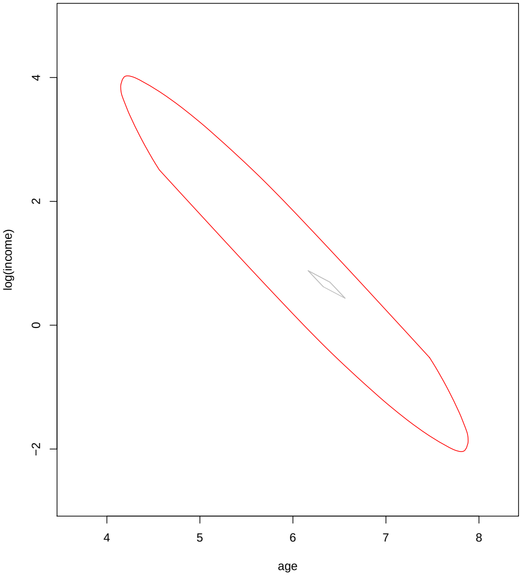
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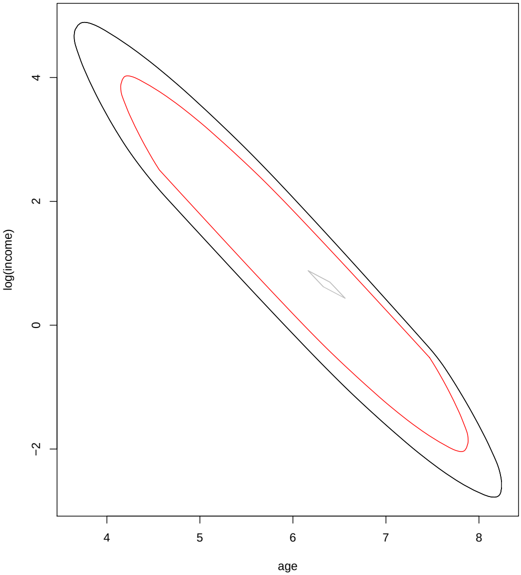

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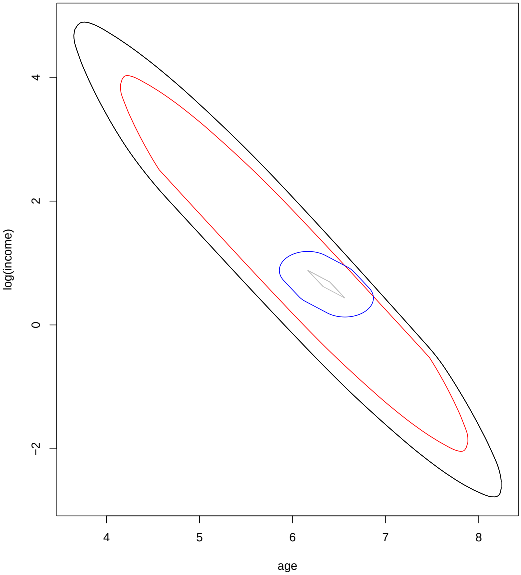
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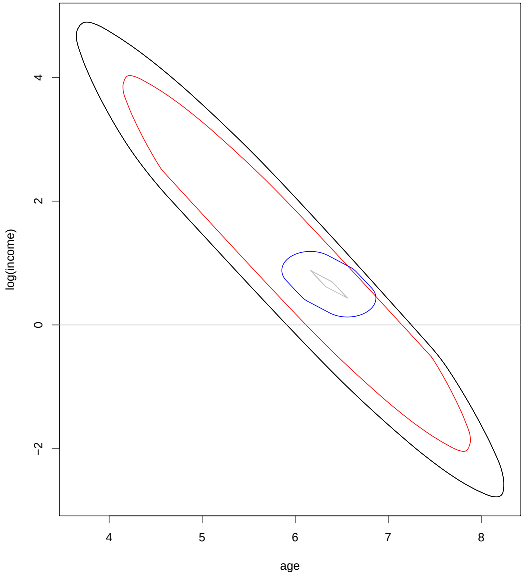

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