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Depth Functions for Non-Standard Data Using Formal Concept Analysis

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Working Group

Working Group *Foundations of Statistics and their Applications* of Prof. Dr. Thomas Augustin.
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Example for a formal context: "bodies of water"

bodies of water		attributes					
		temporary	*running*	*natural*	*stagnant*	*constant*	*maritime*
objects	canal		✓			✓	
	channel		✓			✓	
	lagoon			✓	✓	✓	✓
	lake			✓	✓	✓	
	maar			✓	✓	✓	
	puddle	✓		✓	✓		
	pond			✓	✓	✓	
	pool			✓	✓	✓	
	reservoir				✓	✓	
	river		✓	✓		✓	
	rivulet		✓	✓		✓	
	runnel		✓	✓		✓	
	sea			✓	✓	✓	✓
	stream		✓	✓		✓	
	tarn			✓	✓	✓	
	torrent		✓	✓		✓	
	trickle		✓	✓		✓	

Figure: Formal Context copied from Wikipedia (accessed 25.01.2024)
`https://en.wikipedia.org/wiki/Formal_concept_analysis`

Example of a formal context

Example of a formal context

Example of a formal context

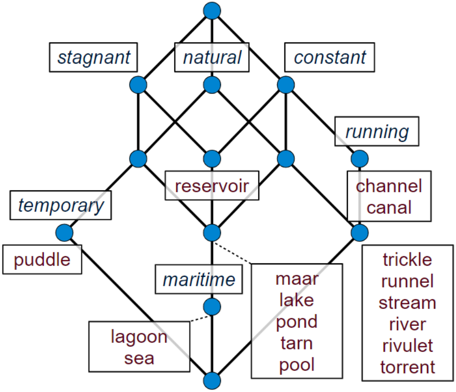

Figure: Lattice copied from Wikipedia (accessed 25.01.2024)
`https://en.wikipedia.org/wiki/Formal_concept_analysis`

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Definition (Formal Context)

A **formal context** is given by a triple $\mathbb{K} = (G, M, I)$. G corresponds to the set of **objects**, M to the set of **attributes** and I defines a binary relation between G and M.[1]

The **derivation operators**

$$\psi\colon 2^G \to 2^M, A \mapsto A' := \{m \in M \mid \forall g \in A\colon gIm\},$$
$$\varphi\colon 2^M \to 2^G, B \mapsto B' := \{g \in G \mid \forall m \in B\colon gIm\}.$$

We call the set $\varphi \circ \psi(2^G)$ the set of extents and $\psi \circ \varphi(2^G)$ the set of intents.

[1] see Ganter et.al. (2012)

This gives us **family of implications** which describes the extent set (and with this also the intent set) completely. Let $A, B \subseteq G$. We say premise A implies conclusion B iff

$$\psi \circ \varphi(A) \supseteq \psi \circ \varphi(B).$$

We denote this by $A \rightarrow B$.

Summary:

$$\text{Data Set} \xrightarrow[\text{method}]{\text{scaling}} \text{Formal Context} \xleftrightarrow{\varphi,\psi} \text{Extent Set} \xleftrightarrow{\varphi,\psi} \text{Family of Implications}$$

A formal context for partial orders

Definition

Let M be a set. Then $(M, \leq)$ is a partial order if and only if for all $a, b, c \in M$

1. Reflexivity: $a \leq a$,
2. Antisymmetry: if $a \leq b$ and $b \leq a$ then a and b are the same element, and
3. Transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$

holds.

A formal context for partial orders

A formal context for partial orders

Let $\mathcal{P}$ be the set of all partial orders on $\mathcal{X} = \{x_1, \dots, x_n\}$ with $n \in \mathbb{N}$.

The formal context $\mathbb{K}$ is given by

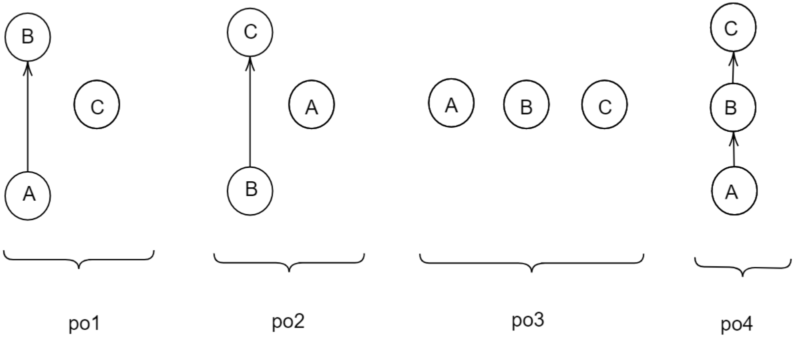
1. $G = \mathcal{P}$
2. $M = \underbrace{\{\text{“}x_i \leq x_j\text{”} \mid i, j = 1, \dots, n,\ i \neq j\}}_{=:M_{\leq}} \cup \underbrace{\{\text{“}x_i \not\leq x_j\text{”} \mid i, j = 1, \dots, n,\ i \neq j\}}_{:=M_{\not\leq}}$
3. $I = \{(g, m) \in G \times M \mid m \text{ is true for } g\}$

This corresponds to the closure operator which maps each subset $\{g_1, \dots, g_m\} \subseteq \mathcal{P} = G$, $m \in \mathbb{N}$ to

$$\{g \in \mathcal{P} \mid \cap_{i=1}^m g_i \subseteq g \subseteq \cup_{i=1}^m g_i\}.$$

A formal context for partial orders

Let $\mathcal{X} = \{A, B, C\}$. Consider the set {po1, po2} and its implications.



We obtain

1. {po1, po2} implies {po1, po2, po3}, but
2. {po1, po2} does not imply {po4}.

Let $\mathcal{P}$ be the set of posets on M. Then with the above formal context we get the mapping

$$\varphi \circ \psi = \gamma \colon \begin{array}{l} 2^{\mathcal{P}} \to 2^{\mathcal{P}} \\ P \mapsto \left\{ p \in \mathcal{P} \mid \bigcap_{\tilde{p} \in P} \tilde{p} \subseteq p \subseteq \bigcup_{\tilde{p} \in P} \tilde{p} \right\}. \end{array}$$

Depth Function

Depth Functions measure **centrality** and **outlingless** of a data point with respect to a data cloud or an underlying distribution.

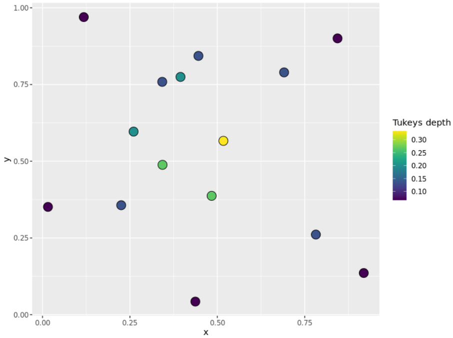


Figure: Tukey depth
15 randomly drawn points

Depth Functions measure **centrality** and **outlingless** of a data point with respect to a data cloud or an underlying distribution.

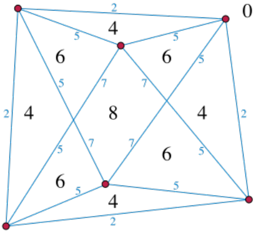


Figure: Simplicial Depth
(see `https://en.wikipedia.org/wiki/Simplicial_depth`, visited: 20.10.23)

Depth Function: Properties

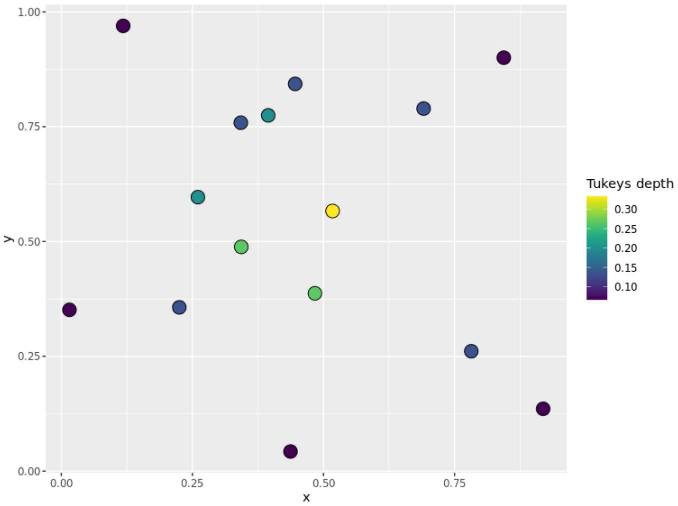


Figure: Tukey depth

Let $\mathcal{F}$ be a set of probability measures on $\mathbb{R}^d$ with $d \in \mathbb{N}$. Let

$$D : \mathbb{R}^d \times \mathcal{F} \to \mathbb{R}_{\geq 0}$$

be bounded, then D is[2]

1. Affine invariance: The depth function is invariant under change of the coordinate system.
2. Maximality at center: If the probability function has a unique center then the depth function has its maximum value at this center.
3. Monotonicity relative to deepest point: The depth function decreases with respect to the value with the maximal depth.
4. Vanishing at infinity: The depth function converges to zero if the norm of the point sequence converges to infinity.
5. Quasiconcavity: For every $\alpha \geq 0$ the set consisting of a depth values larger than α is a convex set.

[2] see Zou et.al. (2000) and Mosler (2013)

Non-Standard Data

Non-Standard Data

Non-Standard Data summarizes all data types that are not given in standard statistical data format.
$\rightarrow$ no metric or other presupposed data structure is imposed on the observations/data.

Examples:

1. The set of partial orders (e.g. comparing ml algorithms or food)
2. Mixed (numeric + nominal + . . .) data (e.g. observing spatial observations together with marks like age, education, crime, ...)

Overall Aim:

Define a Depth Function and Resulting Statistics for Non-Standard Data

$\rightarrow$ Formal Concept Analysis

Adaptation of Depth Functions to Data Represented via FCA

What does centrality mean in this context?

General Definition of Depth Functions on FCA

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Definition

We define a depth function using formal concept analysis[3] by

$$D_G : G \times \varkappa_G \times \mathbb{P}_G \to \mathbb{R}_{\geq 0}$$

for a

1. fixed set of objects G and
2. a set of formal contexts $\varkappa_G \subseteq \{\mathbb{K} \mid G \text{ is object set of } \mathbb{K}\}$.
3. $\mathbb{P}_G$ is a set of probability measures on G defined on a σ-field which contains all extent sets of the corresponding formal contexts of $\varkappa_G$.

[3] see Blocher et.al. (2023b)

What does centrality mean in this context?

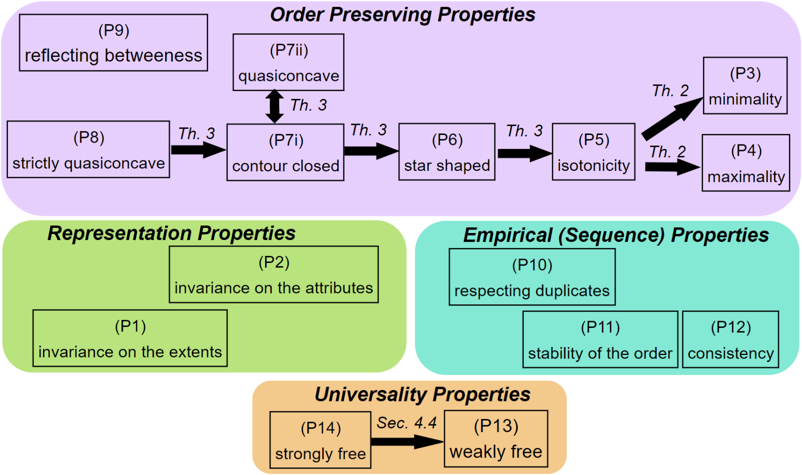

Figure: This figure can be found in Blocher et.al. (2023b).

Adaptation to Data Represented via FCA:

Concrete definition of depth functions using FCA representation?

From the Simplicial Depth to the Union-Free Generic Depth

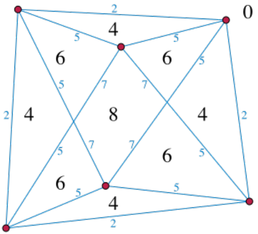

Figure: Simplicial Depth
(see `https://en.wikipedia.org/wiki/Simplicial_depth`,
visited: 20.10.23)

From the Simplicial Depth to the Union-Free Generic Depth

- What does it mean that one object lies in another object?

- What is the correspondence to the vertices defining a triangle?

From the Simplicial Depth to the Union-Free Generic Depth

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Definition

The *union–free generic* family of implications, $\mathcal{U}_{\mathbb{K}}$, for a formal context $\mathbb{K}$ consists of implications $A \rightarrow B$ for which the following is true:

1. they are non–trivial (deleted implications of the form $A \rightarrow A$),
2. they have a minimal premise and a maximal conclusion (deleted implications of the form $A \rightarrow B$ if there exists $\tilde{A} \subsetneq A$ such that $\tilde{A} \rightarrow B$ or $\tilde{B} \subsetneq B$ with $A \rightarrow \tilde{B}$), and
3. cannot be constructed by union from other implications (deleted implications $A \rightarrow B$ if there is a family of implications $(A_i \rightarrow B_i)_{i \in I}$ with $A_i \subsetneq A$ for all $i \in I$ and $A = \cup_i A_i$ and $B = \cup_i B_i$ is true.[a]

[a] Compare this definition to the term proper.

From the Simplicial Depth to the Union-Free Generic Depth

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Definition

Let

- G be a set.
- κ_G be a set of formal contexts with object set G. Moreover, for all $\mathbb{K} \in \kappa_G$ there exists a unique set of union-free generic premises $\mathcal{U}_{\mathbb{K}}$ that completely describes the corresponding closure operator.
- $\gamma_{\mathbb{K}}$ be the closure operator on G corresponding to $\mathbb{K}$.
- $\mathcal{P}_G$ gives a set of probability measures on G.

Then the union-free generic depth is defined as

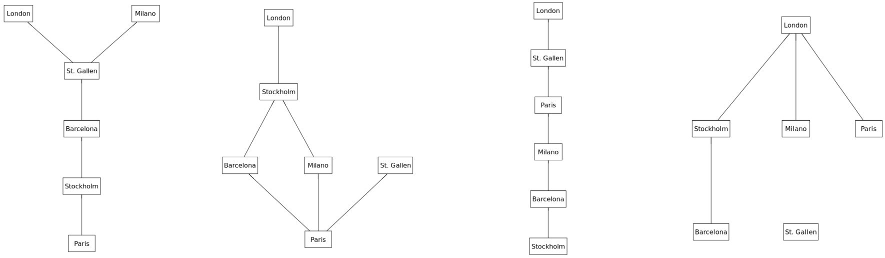
$$D: \begin{array}{l} G \times \kappa_G \times \mathcal{P}_G \to \mathbb{R}_{\geq 0}, \\ (g, \mathbb{K}, P) \mapsto \sum_{j=1}^{\infty} \frac{1}{C_j} P(g \in \gamma_{\mathbb{K}}(X^j) \mid X^j \in \mathcal{U}_{\mathbb{K}}) \end{array}$$

with $X^j = \{X_1, \ldots, X_j\}$ where $X_1, \ldots, X_j \sim P$. Moreover, $C_j \in]0, \infty[$ for all $j \in \mathbb{N}$. We set $\sum_{\emptyset} = 0$ and $P(A \mid B) = 0$ for $P(B) = 0$.

Example: Partial Orders
Comparing Colleges Preferences

We used the data set cemspc of the R-package prefmod, see Dittrich et. al.(1998).

- survey of 303 students (the answer of 63 students fulfils the partial order structure – antisymmetric and transitive – and consists of none NA's)
- examines the student's preferences of 6 universities (London, Paris, Milano, St.Gallen, Barcelona and Stockholm).
- paired comparisons with possible undecided answers



Example: Partial Orders
Comparing Colleges Preferences

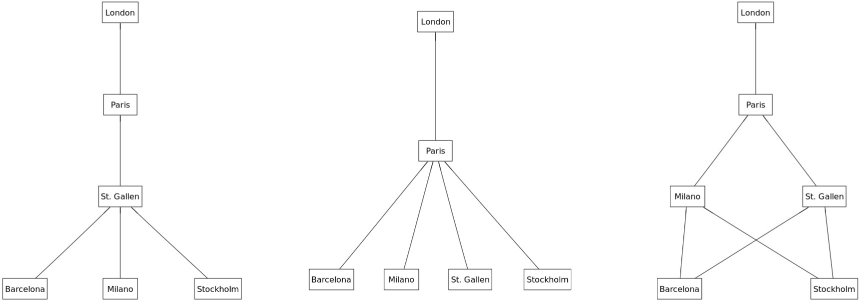


Figure: The partial orders with the highest unimodal ufg depth values (from left to right the depth values: $0.90, 0.88, 0.8$).

Example: Partial Orders
Comparing Colleges Preferences

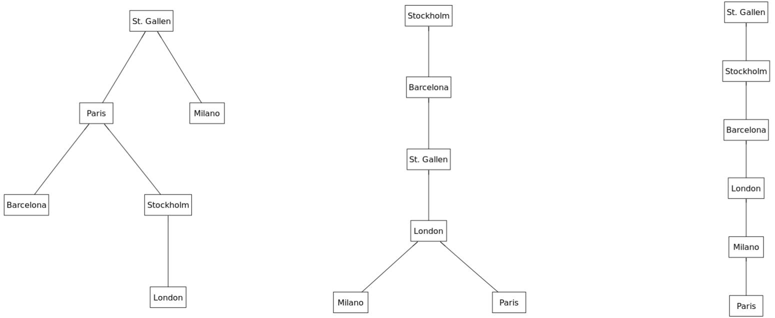


Figure: The partial orders with lowest unimodal ufg depth value 0.11.

Example: Partial Orders
Comparing ML Algorithms

Comparison of Machine Learning Algorithms[5]

- Data Sets: 80 classification problems from OpenML.
- ML Algorithms: Random Forests (RF), Decision Tree (CART), Logistic regression (LR), L1-penalized logistic regression (Lasso) and k-nearest neighbours(KNN).
- Performance Measures: area under the curve, F-score, predictive accuracy and Brier score.

$\Rightarrow$ We obtain 80 posets

[5] see Blocher et.al. (2023a,c)

Example: Partial Orders Comparing ML Algorithms

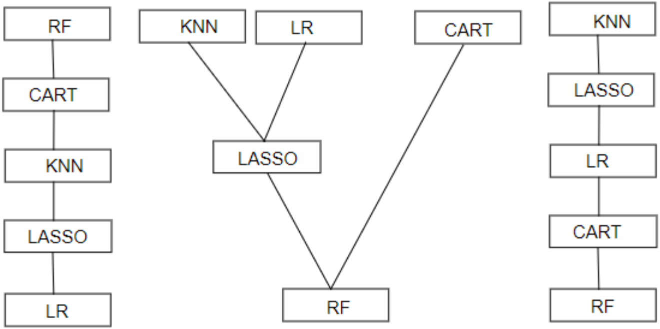


Figure: OpenML based on all four performance measures: Poset with maximal depth based on all possible posets is plotted on the left. The poset with minimal ufg depth restricted to the observed one can be seen in the middle. The poset on the right denotes the poset with minimal depth value based on all possible posets.[6]

[6] see Blocher et.al. (2023c)

Adaptation to Data Represented via FCA:

And what about inference?

Here, we compare the depth function evaluated based on two different (empirical) distributions: One is F and the other G.

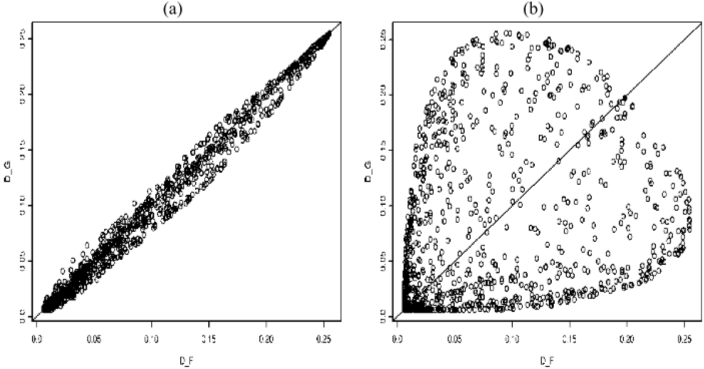


FIG. 2. *DD plots of* (a) *identical distributions and* (b) *location shift.*

Figure: This figure can be found in Li et.al. (2004).

Open Questions and Discussion

Open Questions and Discussion

- Other kind of non-standard data?
- Where could formal concept analysis be used in psychology?
- How to define a one-sample test or regression?
- ...

References I

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Definition of the Simplicial Depth

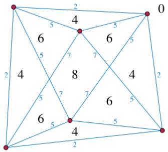
Definition

Let $(\mathbb{R}^d, \mathcal{B})$ with $d \in \mathbb{N}$ be a measurable space and let $\mathcal{F}_{\mathbb{R}^d}$ be a set of probability measures such that $(\mathbb{R}^d, \mathcal{B}, \Pr)$ defines a probability space for each $\Pr \in \mathcal{F}_{\mathbb{R}^d}$. Then the simplicial depth[7] is given by

$$D : \mathbb{R}^d \times \mathcal{F}_{\mathbb{R}^d} \to [0,1], (x, \Pr) \mapsto \Pr(x \in S^d[X_1, \dots, X_{d+1}])$$

with

- $X_1, \dots, X_{d+1}$ independent and identically distributed random variables from Pr, and
- $S^d[X_1, \dots, X_{d+1}]$ being the set of points that are lie in the convex closure of $\{X_1, \dots, X_{d+1}\}$.



[7]see Liu (1990)

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Definition

Let $\mathcal{M}$ be the set of probability measures on $\mathcal{P}$ equipped with $2^{\mathcal{P}}$ as σ-field. The *union-free generic (ufg for short) depth on posets*[8] is given by

$$D\colon \begin{array}{l} \mathcal{P} \times \mathcal{M} \to [0,1] \\ (p,\nu) \mapsto \begin{cases} 0, & \text{if for all } S \in UFG\colon \prod_{\tilde{p}\in S} \nu(\{\tilde{p}\}) = 0 \\ c\sum_{S\in UFG\colon p\in\gamma(S)} \prod_{\tilde{p}\in S} \nu\left(\{\tilde{p}\}\right), & \text{else} \end{cases} \end{array}$$

with $c = \left(\sum_{S\in UFG} \prod_{\tilde{p}\in S} \nu_n\left(\{\tilde{p}\}\right)\right)^{-1}$.

[8] see Blocher et.al. (2023a,c)