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Statistical Models for Partial Orders based on Data Depth and Formal Concept Analysis

IPMU 2022

Motivation

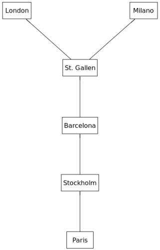
In the most approaches

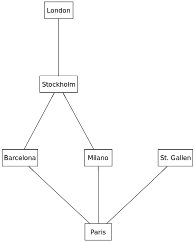
- **incompleteness** of a partial order stems form **missing knowledge/data**, e.g. an unresolved conflict in values or goals (e.g. Stewart 2020).

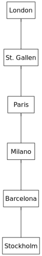
→ an explicit missing mechanism is assumed and included in the model (e.g. by the distance measure used)

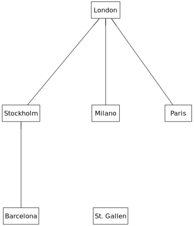
In what follows

- an observed **incomparability** is understood as a **precise observation** and we consider a depth function instead of a distance measure.









Motivation

This distinction is to be represented in the construction of the stochastic model via the set of partial order.

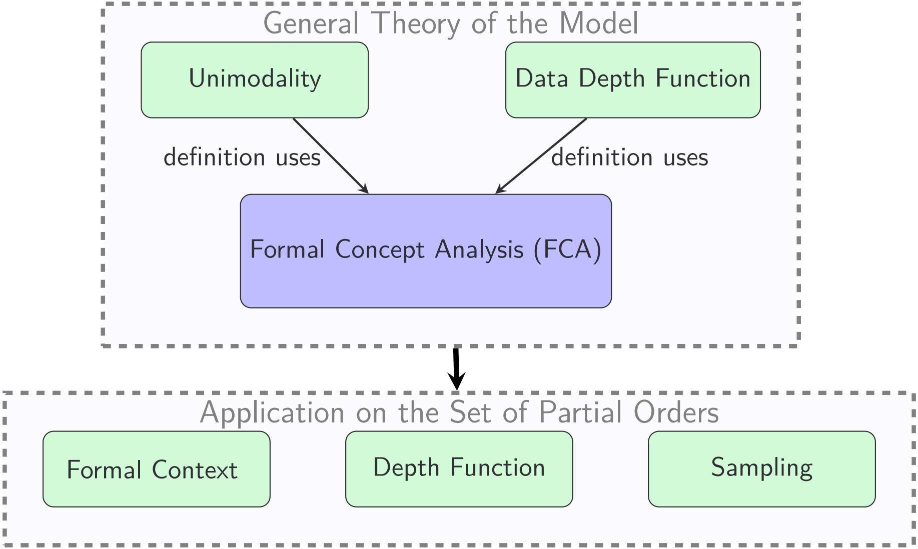
Thus, we consider the statistical model

$$P(X = x) = C_\lambda \cdot \Gamma\left(\lambda \cdot \left(1 - D^\mu\left(x\right)\right)\right)$$

with

1. C_λ normalizing constant,
2. $\Gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ (weakly decreasing) decay function,
3. $\mu \in \mathcal{P}$ location parameter and $\lambda \in \mathbb{R}_{>0}$ a scale parameter and
4. D^μ a depth function that is maximal at partial order μ.

Overview



General Theory of the Model
Unimodality
Data Depth Function
definition uses
definition uses
Formal Concept Analysis (FCA)
Application on the Set of Partial Orders
Formal Context
Depth Function
Sampling

Content

1. Formal Concept Analysis
   (i) Introduction to Formal Concept Analysis
   (ii) Introduction to Depth Functions
   (iii) Definition of Unimodality
2. Application on the Set of Partial Orders
   (i) Formal Context
   (ii) Depth Function
   (iii) Sampling

Formal Concept Analysis

Representation of a data set as cross table

		a	b	c	d	e	f	g	h	i
1	Leech	×	×					×		
2	Bream	×	×					×	×	
3	Frog	×	×	×				×	×	
4	Dog	×		×				×	×	×
5	Spike – weed	×	×		×		×			
6	Reed	×	×	×	×		×			
7	Bean	×		×	×	×				
8	Maize	×		×	×		×			

Figure 1.1 Context of an educational film "Living Beings and Water". The attributes are: a: needs water to live, b: lives in water, c: lives on land, d: needs chlorophyll to produce food, e: two seed leaves, f: one seed leaf, g: can move around, h: has limbs, i: suckles its offspring.

Graphic is taken from Ganter and Wille 2012, p.18

Formal Concept Analysis: Scaling Method

Convert a non-binary attribute into a set of binary attributes by using

$$t : V \rightarrow 2^{M_{new}}$$

where V is the set of non-binary values and M_{new} is the set of binary attributes. Note that the binary relation $\tilde{I}$ (given by the crosses) must also be adjusted accordingly.

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For example:
Let $g_1, \ldots, g_n \in \mathbb{R}^d, d \in \mathbb{N}$ be observations. Then a possible scaling method is

$$t : \mathbb{R}^d \to 2^{\{\text{ set of all half–spaces}\}},\ x \mapsto \{H \mid H \text{ half–space in } \mathbb{R}^d\}$$

and we say $(g_j, H) \in \tilde{I}$ iff g_j lies in half–space H.

Formal Concept Analysis: Formal Context and Deviation Operators

Definition

A **formal context** is given by a triple $\mathbb{K} = (G, M, I)$. G is the set of **objects**, M is the set of **attributes** and I defines a binary relation between G and M.

Formal Concept Analysis: Formal Context and Deviation Operators

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The **derivation operators**

$$\psi \colon 2^G \to 2^M, A \mapsto A' := \{m \in M \mid \forall g \in A \colon gIm\},$$
$$\varphi \colon 2^M \to 2^G, B \mapsto B' := \{g \in G \mid \forall m \in B \colon gIm\}$$

give us the **closure operator**

$$\psi \circ \varphi : 2^G \to 2^G, A \mapsto A''.$$

This gives us

1. the **closure system** $\{A'' \mid A \subseteq G\}$ on G which describes the formal context

and

1. a **family of implications** which describes the closure system completely. Let $A, B \subseteq G$. We say premise A implies conclusion B iff

$$\psi \circ \varphi(A) \supseteq \psi \circ \varphi(B).$$

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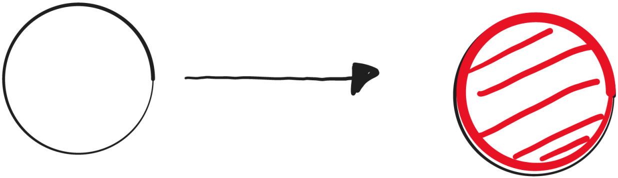

Formal Concept Analysis: Closure System and Implications

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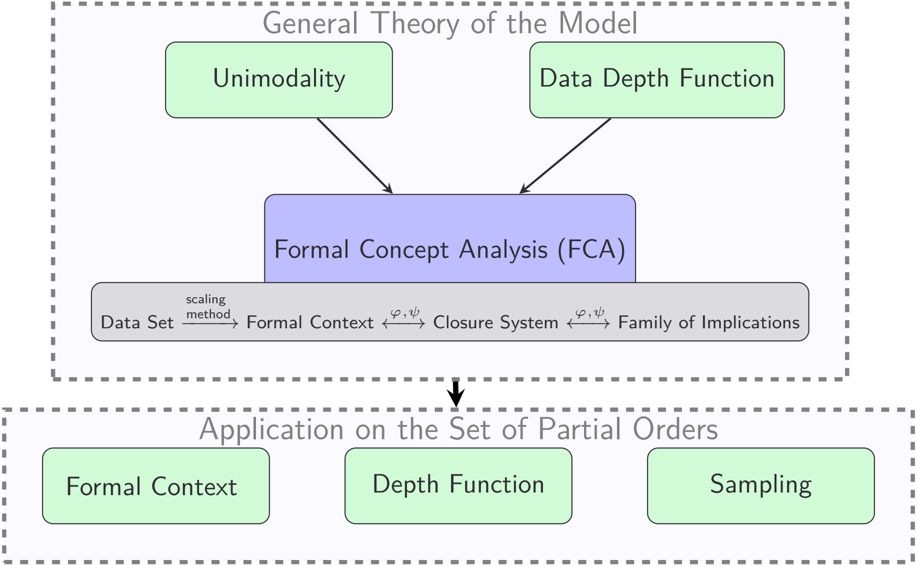
Summary:

$$\text{Data Set} \xrightarrow{\substack{\text{scaling}\\ \text{method}}} \text{Formal Context} \xleftrightarrow{\varphi,\psi} \text{Closure System} \xleftrightarrow{\varphi,\psi} \text{Family of Implications}$$

For further readings we refer to Ganter and Wille 2012

Overview



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scaling method
Formal Context
φ, ψ
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Depth Functions

Measures **centrality** and **outlingless** of a data point with respect to a data cloud or an underlying distribution

[1] https://de.wikipedia.org/wiki/Box-Plot (visited: 20.10.21)
[2] https://link.springer.com/article/10.1007/s10994-015-5524-x (visited: 20.10.21)
[3] https://en.wikipedia.org/wiki/Simplicial_depth (visited: 20.10.21)

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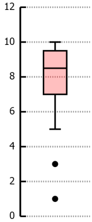

Figure: Quantiles[1]

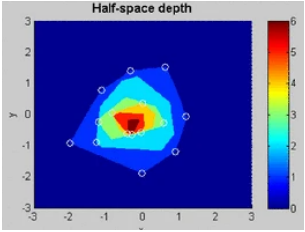

Figure: Halfspace Depth[2]

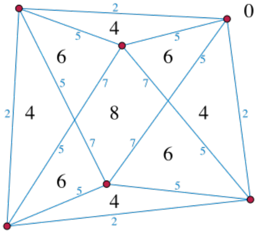

Figure: Simplicial Depth[3]

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Depth Functions: Selection of Properties in $\mathbb{R}^d$

- Affine Invariance
- Monotonicity Relative to the Deepest Point, Quasi–Concavity, ... **(Unimodality)**
- Vanishing at Infinity
- Computability

[3] For further readings, we refer to Zou and Serfling 2000 and Chen et al. 2015

Combining Formal Concept Analysis and Depth Function

- The depth function based on a formal context has as input the set of objects and is defined as $f : G \to \mathbb{R}_{\geq 0}$.
- Approach to obtain a depth function: Use the different representations of a data set given by FCA and define the statistical depth function based on this representation.
- Transfer the properties of $\mathbb{R}^d$ for statistical depth functions to general depth functions defined by a formal context. Here, we restrict to **unimodality** property.

Unimodality in the context of Formal Concept Analysis

A function respects the structure of the family of implications given by a formal context.

Definition

Let $\mathbb{K} = (G, M, I)$ be a formal context and let $f : H \longrightarrow \mathbb{R}_{\geq 0}$ with $H \subseteq G$ be a (depth, probability) function.

Then f is called **unimodal** if for every finite set of objects $\{g_1, \ldots, g_n\} \subseteq H$ with $\{g_1, \ldots, g_{n-1}\}$ implying g_n we have

$$f(g_n) \geq \min\{f(g_1), \ldots f(g_{n-1})\}$$

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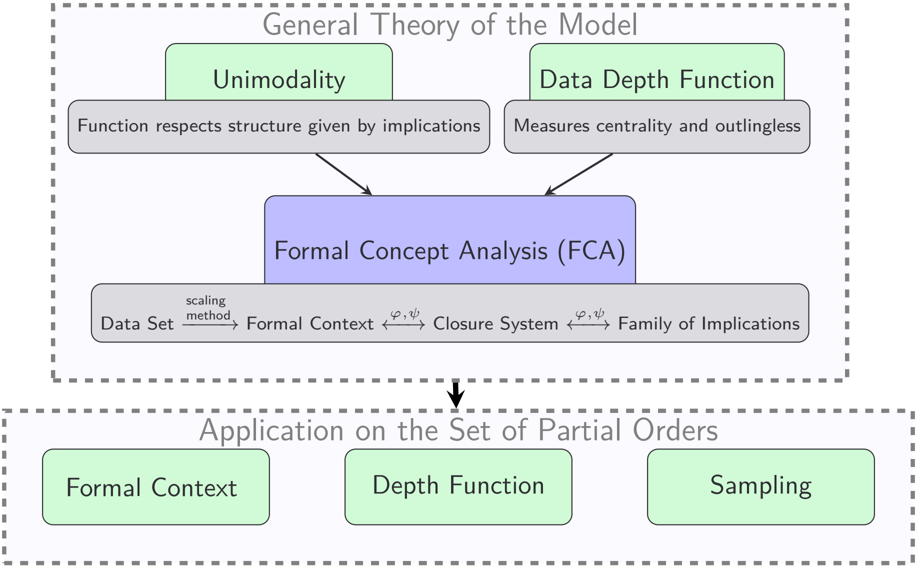
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Application on the Set of Partial Orders

For now on, let $\mathcal{P}$ be the set of all partial orders on $\mathcal{X} = \{x_1, \ldots, x_n\}$ with $n \in \mathbb{N}$.

The formal context $\mathbb{K}$ is given by

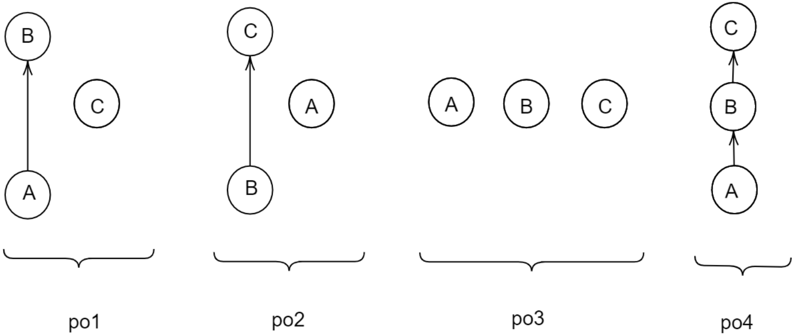
1. $G = \mathcal{P}$
2. $M = \underbrace{\{\text{“}x_i \leq x_j\text{”} \mid i,j = 1, \ldots, n,\ i \neq j\}}_{=:M_{\leq}} \cup \underbrace{\{\text{“}x_i \not\leq x_j\text{”} \mid i,j = 1, \ldots, n,\ i \neq j\}}_{:=M_{\not\leq}}$

This corresponds to the closure operator which maps each subset $\{g_1, \ldots, g_m\} \subseteq \mathcal{P} = G$, $m \in \mathbb{N}$ to

$$\{g \in \mathcal{P} \mid \cap_{i=1}^{m} g_i \subseteq g \subseteq \cup_{i=1}^{m} g_i\}.$$

Application on the Set of Partial Orders: Formal Context

Let $\mathcal{X} = \{A, B, C\}$. Consider the set {po1, po2} and its implications.



We obtain

1. {po1, po2} implies {po1, po2, po3}, but
2. {po1, po2} does not imply {po4}.

Application on the Set of Partial Orders: Generalized Localized Tukeys Depth Function

Definition

Let $\mathbb{K} = (G, M, I)$ be a formal context, then the **generalized localized Tukey's depth** function (see Schollmeyer 2017) is given by

$$\mathscr{T}^{\mu}(g) := 1 - \frac{\max\limits_{\substack{m \in M \setminus \Psi(\{g\}),\\ \mu I m}} |\Phi(\{m\})|}{|G|}.$$

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
$\mu = g_1$	×		×	×	×		×	×		×	
g_2		×		×		×					
g_3	×	×		×	×						
g_4	×			×			×				
g_5	×						×		×		×

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$\mu = g_1$	×		×	×	×		×	×		×	
g_2		×		×		×					
g_3	×	×		×	×						
g_4	×			×			×				
g_5	×						×		×		×

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For the formal context with $G = \mathcal{P}$ and $M = M_{\leq} \cup M_{\not\leq}$ we get

$$\mathscr{T}^{\mu}(g) = 1 - \max\left\{ \max_{(p,q) \in \mu \setminus g} \alpha_{p,q}, \max_{(p,q) \in g \setminus \mu} \beta_{p,q} \right\} \text{ with } \alpha_{p,q}, \beta_{p,q} \in [0, 1],$$

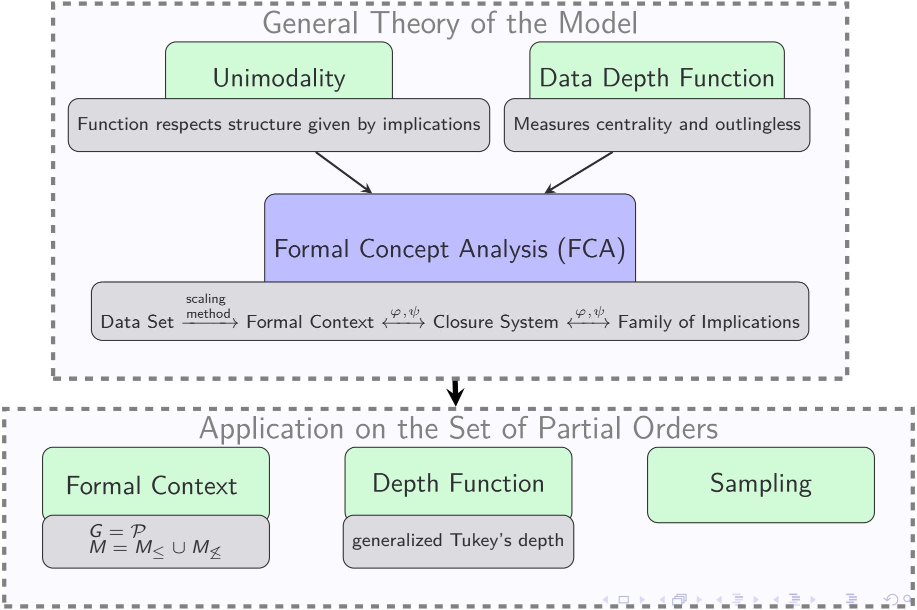
$\rightarrow$ Note that $\alpha_{p,q}$ and $\beta_{p,q}$ are constant and not dependend of p or q.

Application on the Set of Partial Orders: Generalized Localized Tukeys Depth Function

Regarding reweighted version of $\alpha_{p,q}$ and $\beta_{p,q}$:

- $\alpha_{p,q} \propto |\{r \in \mathcal{X} \mid p \leq_\mu r \leq_\mu q\}|$
- $\beta_{p,q} \propto |\{r \in \mathcal{X} \mid p \wedge_\mu q \leq_\mu r \leq_\mu p \vee_\mu q\}| - 1$

Overview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Recall ...

the statistical model

$$P(X = x) = C_\lambda \cdot \Gamma\left(\lambda \cdot \left(1 - D^\mu\left(x\right)\right)\right)$$

with

1. C_λ normalizing constant,
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Application on the Set of Partial Orders: Sampling

Acceptance-Rejection Method:

- **Step 1**: Draw systematically a partial order

$$P_{algo\_select}(g) = |lext(g)| \cdot 2^{|g|-|reduc(g)|} \cdot \left(n!2^{n(n-1)/2}\right)^{-1}$$

- **Step 2**: Compute the acceptance probability

$$acc(g) = f(g) \cdot \left(P_{algo\_select}(g) \cdot n!2^{n(n-1)/2}\right)^{-1}.$$

- **Step 3**: Sample uniformly a value between $[0, 1]$ and if its lower than $acc(g)$ accept the partial order given in Step 1.

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$G = \mathcal{P}$
$M = M_{\leq} \cup M_{\not\leq}$

Depth Function
generalized Tukey's depth

Sampling
Acceptance Rejection Methode

Outlook

- How to estimate a statistical model from a sample?
- Inference?
- How to deal with non-transitive or/and cyclic observations? How to deal with NA's?

References

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