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Statistical Models for Partial Orders based on Data Depth and Formal Concept Analysis

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Motivation

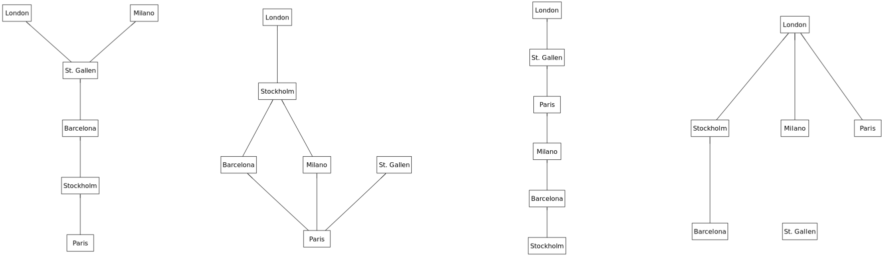
In the most approaches known to the presenter

- incompleteness of a partial order stems form missing knowledge/data, e.g. an unresolved conflict in values or goals (e.g. Stewart 2020).

$\rightarrow$ an explicit missing mechanism is assumed.

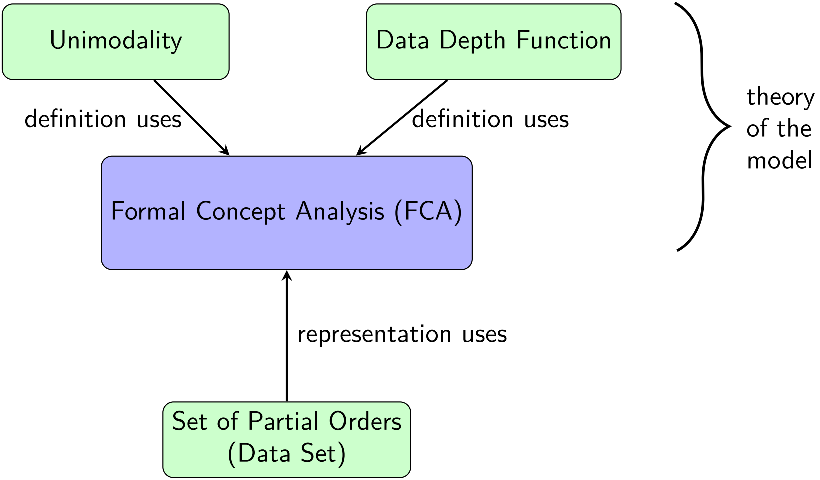
In what follows

- an observed incomparability is understood as a precise observation.



these partial orders are observations of the dataset cemspc in the R-package prefmod

Overview


Unimodality
Data Depth Function
definition uses
definition uses
theory of the model
Formal Concept Analysis (FCA)
representation uses
Set of Partial Orders (Data Set)

Content

1. Formal Concept Analysis
   (i) Depth Functions
   (ii) Definition of Unimodality
2. Application on the Set of Partial Orders
3. References

Formal Concept Analysis

Representation of a data set as cross table

		a	b	c	d	e	f	g	h	i
1	Leech	×	×					×		
2	Bream	×	×					×	×	
3	Frog	×	×	×				×	×	
4	Dog	×		×				×	×	×
5	Spike – weed	×	×		×		×			
6	Reed	×	×	×	×		×			
7	Bean	×		×	×	×				
8	Maize	×		×	×		×			

Figure 1.1 Context of an educational film "Living Beings and Water". The attributes are: a: needs water to live, b: lives in water, c: lives on land, d: needs chlorophyll to produce food, e: two seed leaves, f: one seed leaf, g: can move around, h: has limbs, i: suckles its offspring.

Graphic is taken from Ganter and Wille 2012, p.18

Formal Concept Analysis: Scaling Method

Transforming a non-binary attribute into a set of binary attributes by the use of an injective function

$$t : V \to 2^{M_{new}}$$

with V being the set of non-binary values and M_{new} the set of binary attributes.

Formal Concept Analysis: Scaling Method

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For example:
We observe observation in $\mathbb{R}^d, d \in \mathbb{N}$. Then a possible scaling method is

$$t : \mathbb{R}^d \to 2^{\{\text{ set of all half–spaces}\}}, \; x \mapsto \{H \mid x \text{ lies in half–space } H\}$$

Formal Concept Analysis: Formal Context and Deviation Operators

Definition (Formal Context)

A **formal context** is given by a triple $\mathbb{K} = (G, M, I)$. G corresponds to the set of **objects**, M to the set of **attributes** and I defines a binary relation between G and M.

Formal Concept Analysis: Formal Context and Deviation Operators

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The **derivation operators**

$$\psi\colon 2^G \to 2^M, A \mapsto A' := \{m \in M \mid \forall g \in A\colon gIm\},$$
$$\varphi\colon 2^M \to 2^G, B \mapsto B' := \{g \in G \mid \forall m \in B\colon gIm\}$$

give us the closure operator

$$\psi \circ \varphi : 2^G \to 2^G, A \mapsto A''.$$

Formal Concept Analysis: Closure System and Implications

This gives us

1. the **closure system** $\{A'' \mid A \subseteq G\}$ on G which describes the formal context and
2. a **family of implications** which describes the closure system completely. Let $A, B \subseteq G$. We say A implies B iff

$$\psi \circ \varphi(A) \supseteq \psi \circ \varphi(B).$$

For further readings we refer to Ganter and Wille 2012

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Summary:

$$\text{Data Set} \xrightarrow[\text{method}]{\text{scaling}} \text{Formal Context} \xleftrightarrow{\varphi, \psi} \text{Closure System} \xleftrightarrow{\varphi, \psi} \text{Family of Implications}$$

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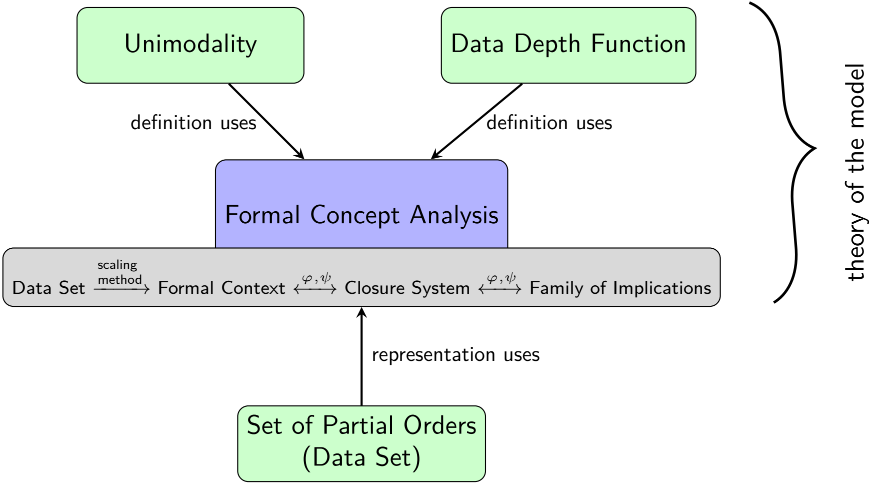

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Depth Functions

Measures centrality and outlingless of a data point with respect to a data cloud or an underlying distribution

[1] https://de.wikipedia.org/wiki/Box-Plot (visited: 20.10.21)
[2] https://link.springer.com/article/10.1007/s10994-015-5524-x (visited: 20.10.21)
[3] https://en.wikipedia.org/wiki/Simplicial_depth (visited: 20.10.21)

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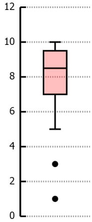

Figure: Quantiles[1]

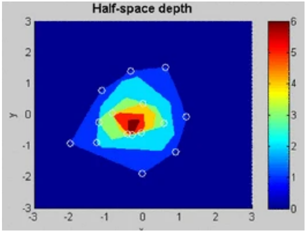

Figure: Halfspace Depth[2]

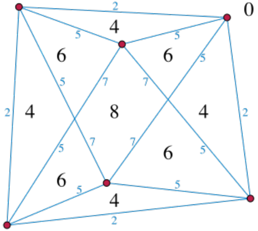

Figure: Simplicial Depth[3]

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Depth Functions: Selection of Properties in $\mathbb{R}^d$

- Affine Invariance
- Monotonicity Relative to the Deepest Point, Quasi–Concavity, ... **(Unimodality)**
- Vanishing at Infinity
- Computability

[3] For further readings, we refer to Zou and Serfling 2000 and Chen et al. 2015

Combining Formal Concept Analysis and Depth Function

- The depth function based on a formal context has as input the set of objects and is defined as $f : G \to \mathbb{R}_{\geq 0}$.
- Approach to obtain a depth function: Use the different representations of a data set given by FCA and define the statistical depth function based on this representation.
- Transfer the properties of $\mathbb{R}^d$ for statistical depth functions to general depth functions defined by a formal context. Here, we discuss the **unimodality** property.

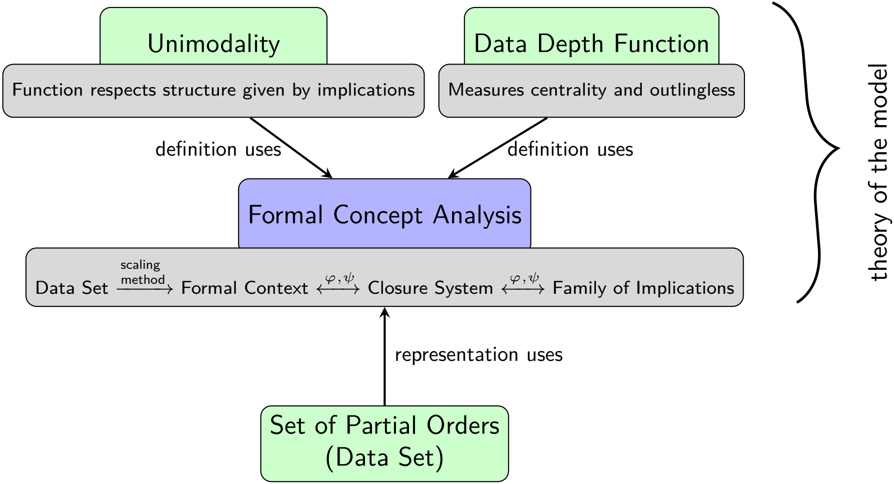
Unimodality in the context of Formal Concept Analysis

A function respects structure of the family of implications given by a formal context

Definition

Let $\mathbb{K} = (G, M, I)$ be a formal context and let $f : H \longrightarrow \mathbb{R}_{\geq 0}$ with $H \subseteq G$ be a (depth, probability) function.
Then f is called **unimodal** if for every finite set of objects $\{g_1, \ldots, g_n\} \subseteq H$ with $\{g_1, \ldots, g_{n-1}\}$ implying g_n we have

$$f(g_n) \geq \min\{f(g_1), \ldots f(g_{n-1})\}$$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Application on the Set of Partial Orders

For now on, let $\mathcal{P}$ be the set of all partial orders on $\mathcal{X} = (x_1, \ldots, x_n)$ with $n \in \mathbb{N}$.

The formal context $\mathbb{K}$ is given by

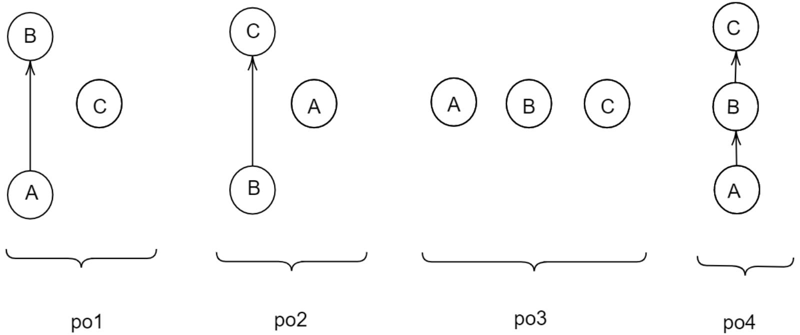
1. $G = \mathcal{P}$
2. $M = \underbrace{\{\text{“}x_i \leq x_j\text{”} \mid i, j = 1, \ldots, n,\ i \neq j\}}_{=:M_{\leq}} \cup \underbrace{\{\text{“}x_i \not\leq x_j\text{”} \mid i, j = 1, \ldots, n,\ i \neq j\}}_{:=M_{\not\leq}}$

This corresponds to the closure operator which maps each subset $\{g_1, \ldots, g_m\} \subseteq \mathcal{P} = G$, $m \in \mathbb{N}$ to

$$\{g \in \mathcal{P} \mid \cap_{i=1}^{m} g_i \subseteq g \subseteq \cup_{i=1}^{m} g_i\}.$$

Application on the Set of Partial Orders

Let $\mathcal{X} = \{A, B, C\}$. Consider the set {po1, po2} and its implications.



We obtain

1. {po1, po2} implies {po1, po2, po3}, but
2. {po1, po2} does not imply {po4}.

Union–free Generic Family of Implications and the resulting Depth Function

Definition (Union–free Generic Family of Implications)

The *union–free generic* family of implications, $\mathcal{UFG}$, consists of implications $A \to B$ for which the following is true:

1. they are non–trivial,
2. they have a minimal premise and a maximal conclusion, and
3. they can not be divided by a further set of implications $\mathcal{I}$ with minimal premise and maximal conclusion. We say an implication $A \to B$ can be divided by a set of implications $\mathcal{I}$ if the union of the premise A equals the union of the premises of the implications in $\mathcal{I}$ and the same holds for the conclusions (the premises in $\mathcal{I}$ do not need to be disjoint, but proper subsets of A).

$\to$ This family of implications is sufficient to describe the corresponding closure system.

Union–free Generic Family of Implications and the resulting Depth Functions

Definition

For a given formal context $\mathbb{K} = (G, M, I)$ we define the **ufg depth function** based on the corresponding union–free generic family of implications $\mathcal{UFG}$ by

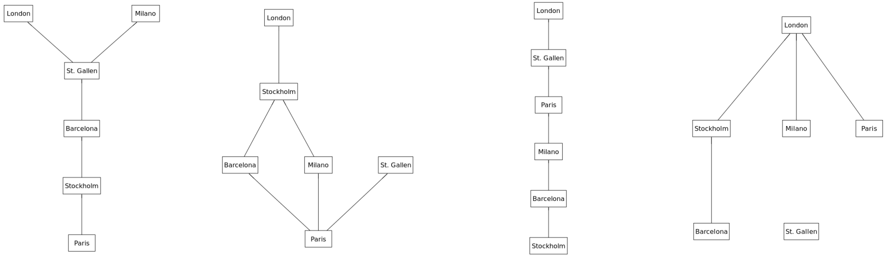
$$D(\cdot, \mathbb{K})\colon G \to [0,1],\ g \mapsto \frac{\#\{A \to B \in \mathcal{UFG} \mid g \in B\}}{\#\{A \to B \in \mathcal{UFG}\}}.$$

$\lightning$ The corresponding depth function is not unimodal. To preserve the unimodal property, each partial order that violates it is assigned the lowest depth value with which the unimodal property still holds.

Example: Community of European management school (paired comparisons with undecided)

We used the data set cemspc of the R-package prefmod, see Dittrich, Hatzinger, and Katzenbeisser 1998.

- survey of 303 students (the answer of 63 students fulfils the partial order structure – antisymmetric and transitive – and consists of none NA's)
- examines the student's preferences of 6 universities (London, Paris, Milano, St.Gallen, Barcelona and Stockholm).
- paired comparisons with possible undecided answers



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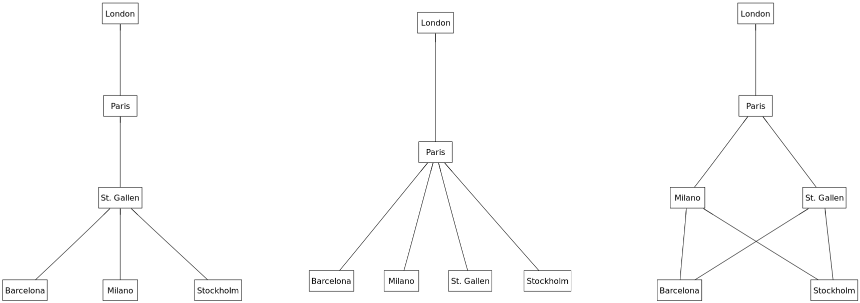

Figure: The partial orders with the highest unimodal ufg depth values (from left to right the depth values: $0.90, 0.88, 0.8$).

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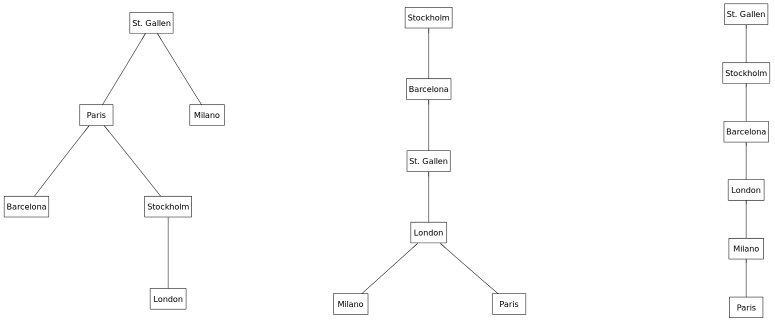

Figure: The partial orders with lowest unimodal ufg depth value 0.11.

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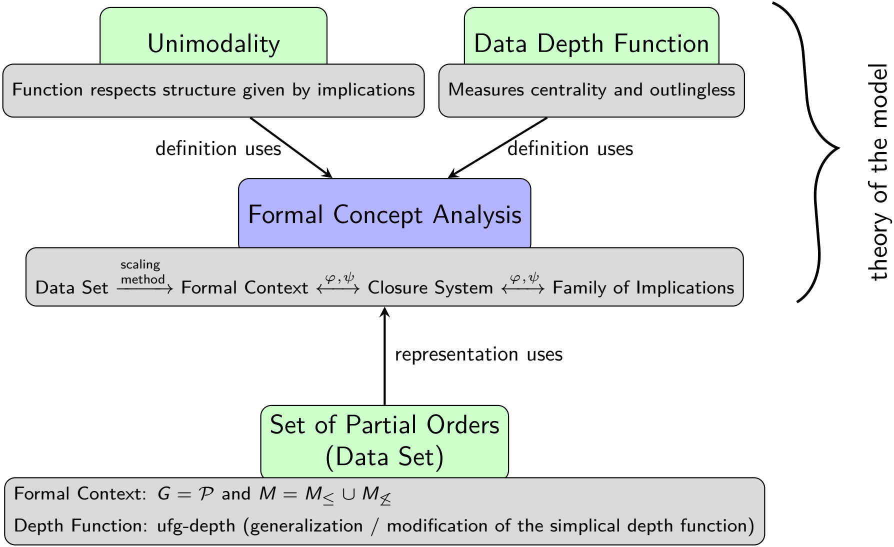

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Formal Context: $G = \mathcal{P}$ and $M = M_{\leq} \cup M_{\not\leq}$
Depth Function: ufg-depth (generalization / modification of the simplical depth function)

Outlook and Discussion

- Based on the this depth function we can a build a statistical model and with an acceptance-rejection method a sampling method can be defined.
- How to deal with non-transitive or/and cyclic observations?
- How to deal with NA's?

References I

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