Dominating countably many forecasts.*

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Abstract

We contrast de Finetti's two criteria for coherence in settings where more than finitely many options are combined into a single option. Coherence$_1$ requires that finitely many previsions cannot be uniformly strictly dominated by abstaining. Coherence$_2$ requires that finitely many probabilistic forecasts cannot be uniformly strictly dominated under Brier score by a rival set of forecasts. Though de Finetti established that these two criteria are equivalent, we show that when previsions/forecasts are based on a merely finitely and not countably additive probability, the second criterion may be extended to permit combining countably infinite sets of options, but not the first criterion. Also, we investigate called-off previsions and called-off forecasts given elements of a partition π, where the called-off previsions/forecasts are based on the conditional probabilities given elements of that partition. We show that each coherence criterion is violated by combining infinitely many called off previsions/forecasts when those conditional probabilities are not conglomerable in the partition π. We show that neither criterion is violated by combining infinitely many called-off previsions/forecasts when conditional expectations are disintegrable in π.

Keywords: Brier score, coherence, conglomerable probability, dominance, finitely additive probability, sure-loss.

1. *Introduction.* Coherence of preference, as de Finetti formulates it (de Finetti, 1974, chapter 3), is the criterion, formulated in either of two ways as explained below in Section 1.2, that a rational decision maker respects *uniform (strict) dominance with respect to a partition* π – hereafter abbreviated π-*UDom*. In section 1.1 we explain the Dominance Principle that de Finetti uses and several conditions that relate it to maximizing expected utility. In Section 1.2 we review de Finetti's two versions of coherence, with a focus on how these allow as coherent preferences based on a finitely additive probability.

1.1 ***Dominance***. Let I index a set of exclusive options and let J index a partition of states. Consider a hypothetical decision problem $\mathcal{O}$ specified by a set of exclusive options $\mathcal{O} = \{O_i : i \in I\}$. Each option O_i is defined as a function from a partition of states, $\pi = \{h_j : j \in J\}$ to real numbers. $O_i(h_j) = r_{ij}$ conveys the information that r_{ij} is the decision maker's cardinal utility for option O_i in state h_j. That is, the quantities r_{ij} are defined up to a common positive linear transformation.

Definition: π-*UDom*. Let O_l and O_m be two options from $\mathcal{O}$. If there exists an $\varepsilon > 0$ such that for each $j \in J$, $r_{lj} > r_{mj} + \varepsilon$, then option O_l uniformly strictly dominates O_m in π.

Uniform strict dominance is stronger than *simple dominance*, which obtains between O_l and O_m when for each $j \in J$, $r_{lj} > r_{mj}$.

Dominance Principle: Let O_l and O_m be options in $\mathcal{O}$. If O_l uniformly (strictly) dominates O_m in some partition π, then O_m is an *inadmissible* choice from $\mathcal{O}$.

The following condition is necessary for the Dominance Principle to be valid with preferences that maximize conditional expected utility.

Act-state independence – no *moral hazard*: We assume that the decision maker has uncertainty about states in π represented by a conditional probability function over (measurable subsets of) π, given an option O_i, $P(\bullet \mid O_i)$. Act-state independence requires that this conditional probability function is independent of each option: for all $i \in I$, $P(\bullet \mid O_i) = P(\bullet)$. It is a familiar observation, as illustrated in Example 1 (below), that in the presence of act-state dependence, even with finite partitions, the Dominance Principle is invalid for a decision maker who maximizes conditional expected utility.

Example 1: (*Dominance versus conditional expected utility when moral hazard is present.*) Consider a binary partition $\Omega = \{\omega_1, \omega_2\}$ and a decision problem $\mathcal{O} = \{O_1, O_2\}$ with two options defined by $O_1(\omega_1) = 1$, $O_1(\omega_2) = 3$, $O_2(\omega_1) = 2$, and $O_2(\omega_2) = 4$. Option O_2 dominates option O_1 in $\pi = \Omega$. But suppose there is moral hazard and the decision maker's conditional probabilities satisfy: $P(\omega_1 \mid O_1) < P(\omega_1 \mid O_2) - 0.5$. Then, contrary to the Dominance Principle, the conditional expected utility of option O_1, given that O_1

obtains, is greater than the conditional expected utility of option O_2, given that O_2 obtains.◊

Consider the following restricted version of the dominance relation.

Definition: *Robust-π-Dom*. Let O_l and O_m be two options from $\mathcal{O}$.

If $inf_j\{r_{lj}\} > sup_j\{r_{mj}\}$, then option O_l *robustly* dominates O_m in π.

When an option robustly dominates another in a partition, it also uniformly strictly dominates it. What is relevant to our analysis here is that when an option O_l robustly dominates another option O_m, then moral hazard does not affect the validity of the Dominance Principle: then O_m is inadmissible if O_l belong to $\mathcal{O}$. That is, when robust dominance obtains in a finite partition, Dominance is valid for an agent who maximizes conditional expected utility regardless moral hazards. As we note below, de Finetti's criterion of incoherence$_1$ for *fair* prices provides an instance of robust dominance, whereas his criterion of incoherence$_2$ for forecasts subject to Brier score does not.

1.2 *Coherence$_1$ and Coherence$_2$*. De Finetti (de Finetti, 1974, chapter 3) formulated two criteria of *coherence* for rational degrees of belief that are based on the Dominance Principle applied to the partition of states that is used to define the options. Coherence$_1$ is formulated for *previsions* (i.e., *fair prices* for buying and selling) of bounded random variables. Coherence$_1$ requires that the sum of payoffs from each *finite* set previsions may not be uniformly strictly dominated in the partition of states by the alternative option of abstaining.

Coherence$_2$ is formulated using Brier-score (squared-error) as a penalty for probabilistic forecasting. Coherence$_2$ requires that the combined (summed) penalty from each *finite* set of the decision maker's probabilistic forecasts is not uniformly strictly dominated in the partition of states by the combined (summed) penalty from any rival set of forecasts for the same variables.

By connecting degrees of belief with decision making in each of these two ways, de Finetti proved that the decision maker's coherent previsions and forecasts are represented by a finitely additive personal probability, $P(\cdot)$, defined below. The decision maker's coherent prevision for a bounded random variable X is her/his P-expected value of X, $E_P[X]$. When the random variable is the indicator for the event F, which following the usual convention we identify with the event itself, then her/his prevision and forecast for event F is her/his personal probability of F, $E_P[F] = P(F)$.

In this paper we consider limitations on the Dominance Principle which arise for coherent preferences that are based on expected values $E_P[\cdot]$ from a finitely additive probability P.

Definitions:

A probability $P(\cdot)$ is *finitely additive* provided that, when events F and G are disjoint, i.e., when $F \cap G = \phi$, then $P(F \cup G) = P(F) + P(G)$. More generally, for a pair of bounded random variables X and Y, finitely additive expected values satisfy

$$E_P(X + Y) = E_P(X) + E_P[Y].$$

A probability is *countably additive* provided that when F_i (i = 1, …) is a denumerable sequence of pairwise disjoint events, i.e., when $F_i \cap F_j = \phi$ if $i \neq j$, then $P(\cup_i F_i) = \sum_i P(F_i)$. More generally, let $\{X_i\}$ be a sequence of bounded variables whose sum $Y = \sum_i X_i$ has a well defined expectation, $-\infty < E_P[Y] < \infty$. Then the countably additive P-expected value satisfies $E_P[Y] = \sum_i E_P[X_i]$.

We call a probability P *merely finitely additive* when P is finitely but not countably additive. Likewise, then its P-expectations are merely finitely additive.

Here we examine when, subject to the Dominance Principle, each of the two senses of coherence can be extended to allow combining countably many bets and/or forecasts into a single act by summing together their countably many outcomes. The two criteria behave differently in this regard when probability is merely finitely additive. We show in *Proposition* 1 of Section 2 (subject to a condition of finite expectations for sums of random variables and for sums of their squares), that a countable sum of Brier scores from a set of coherent_2 unconditional finitely additive probabilistic forecasts cannot be uniformly dominated by any rival set of forecasts. As we explain in Section 2, de Finetti (1949) showed that with a finitely additive probability P, its coherent_1 previsions for elements of a countable partition may be summed together without violating the Dominance Principle if and only if P is countably additive.

In addition to using his two coherence criteria to link coherent unconditional previsions and unconditional forecasts of events to finitely additive unconditional probabilities, de Finetti applied the same two criteria also to called-off previsions and called-off forecasts given an event h. The quantity hX, which equals X when event h obtains and equals 0 otherwise, is the called-off quantity X given event h. De Finetti showed that a coherent called-off prevision/forecast for an event F given an event h requires using the conditional probability P(F | h). We discuss this topic in Section 3.

Central to our findings about when the sum of countably many coherent called-off previsions or forecasts together satisfies the Dominance Principle is the concept of *conglomerability* for a probability P. Conglomerability in a partition $\pi = \{h_1, h_2, \ldots\}$ of conditional expectations $E_P[\,\cdot \mid \pi]$ over the class of (bounded) random variables X is the requirement that the unconditional P-expectation a variable X lies within the range of its conditional P-expectations given elements of π. That is, if for each X,

$$\inf_{j\in J} \{E_P[X \mid h_j]\} \leq E_P[X] \leq \sup_{j\in J} \{E_P[X \mid h_j]\},$$

then P is *conglomerable* in the partition π. For random variables that are indicator functions, conglomerability of P in π entails that the unconditional probability P(F) lies within the range of its conditional probabilities $\{P(F \mid h_i)\}$. When this fails for some event F we say that the conditional probabilities $\{P(F \mid h_i)\}$ are not conglomerable in π.

In *Proposition* 2 we establish that, subject to the Dominance Principle, neither coherence$_1$ nor coherence$_2$ allows taking a countable sum of called-off previsions, or a countable sum of called-off Brier scores, when the decision maker's (finitely additive) probability fails to be conglomerable in a partition π whose elements constitute the conditioning events for the infinitely many called-off quantities. With *Proposition* 3, we establish that when the decision maker's expectations are conglomerable in a partition π, then the Dominance

Principle is not violated by taking countable sums, either for called-off previsions or for called-off forecasts defined with respect to π. In our [1984] we showed that each merely finitely additive probability has conditional probabilities that fail to be conglomerable in some countable partition, whereas each countably additive probability has expectations that are conglomerable in each countable partition. Thus, we arrive at the following conclusions:

- Unless unconditional coherent$_1$ previsions arise from a countably additive probability, the Dominance Principle shows that combining countably many unconditional coherent$_1$ previsions into a single option may be robustly dominated by abstaining.
- However, subject to a condition of finite expectations for sums of random variables and for sums of their squares, Brier score is not similarly affected. Countably many coherent$_2$ unconditional forecasts may be summed together without leading to a violation of the Dominance Principle.
- Unless called-off previsions or called-off forecasts arise from a set of conglomerable conditional probabilities, the Dominance Principle does not allow combining countably many of these quantities into a single option. Hence, only countably additive conditional probabilities satisfy the Dominance Principle when an arbitrary countable set of called-off quantities are summed together.

De Finetti's interest in coherence$_1$ dates from his seminal work (de Finetti, 1937) on subjective probabilities. His interest in coherence$_2$ came later, (see de Finetti,1981), when he recognized that it (but not coherence$_1$) provided also an incentive compatible solution to the problem of mechanism design for eliciting a coherent set of personal probabilities. Specifically, for a forecaster who maximizes subjective expected utility with Brier score as her/his cardinal utility loss function, Brier score is *strictly proper*, which means that her/his uniquely optimal forecast for an event F is her/his personal probability of event F. We discuss propriety of Brier score in section 4 of our paper.

2. *Dominance for countable sums of unconditional previsions/forecasts*.

Let $\{\Omega, \mathcal{B}\}$ be a measurable space, and $\{X_i\}$ a denumerable set of random variables measurable with respect to this space.

Consider, first, coherence$_1$. The decision maker is called upon to provide an unconditional *prevision* or *fair price,* $P(X_i) = p_i$ for each X_i, where he/she is willing to accept the utility payoff $\alpha_i(X_i(\omega) - p_i)$ in state ω. Thus, for $\alpha_i > (<) 0$, the decision maker agrees to buy (respectively, to sell) $|\alpha_i|$-many units of X_i at the price p_i. The real numbers $\{\alpha_i\}$ are selected by an *opponent* after the decision maker has chosen her/his fair prices, $\{p_i\}$, one for each of the X_i. The decision maker's net utility gain **G** in state ω, for a finite number of previsions, p_i (i = 1, …, *n*), when the opponent chooses quantities $\{\alpha_i\}$ is the sum of the separate utility payoffs, $\mathbf{G}(\omega,\{p_i\},\{\alpha_i\}) = \sum_{i=1}^{n} \alpha_i(X_i(\omega) - p_i)$. In de Finetti's scheme, the decision maker is obliged to accept as a single option the finite combination of options each of which he/she judges *fair*.

Coherence_1 is the requirement that the opponent cannot make a *Book* by selecting *finitely* many α_i such that $\mathbf{G}(\omega,\{p_i\},\{\alpha_i\})$ is uniformly strictly dominated in Ω by the decision maker's alternative option to abstain from the market, which action has the constant net utility payoff 0. Dominance asserts that such a finite Book, arising from the finite set of previsions $\{p_i\}$, is inadmissible against the option to abstain.

Coherence_2 addresses a different decision problem where the decision maker is called upon to provide an unconditional forecast $P(X_i) = p_i$ for each random variable X_i, subject to Brier-score squared-error utility payoff, **BS**, expressed as a penalty or utility loss. In state ω the Brier score penalty for a finite number of forecasts, $\{p_i\}$ of variables $\{X_i\}$, $i = 1, \ldots, n$, is the sum: $\mathbf{BS}(\omega, \{p_i\}) = \sum_{i=1}^{n} (X_i(\omega) - p_i)^2$. Coherence_2 is the requirement that there is no set of rival forecasts, $\{q_i\}$ for $\{X_i\}$, such that $\mathbf{BS}(\omega, \{q_i\})$ uniformly strictly dominates $\mathbf{BS}(\omega, \{p_i\})$ in the partition of states, Ω. Dominance asserts that the finite set of $\{p_i\}$ forecasts is inadmissible if there exists a set of dominating rival $\{q_i\}$ forecasts.

De Finetti (1974) established that

(i) A set of previsions are coherent_1 *fair* prices if and only if the same quantities are coherent_2 forecasts, and

(ii) A set of previsions/forecasts are coherent in either sense if and only there exists a finitely additive probability $P(\cdot)$ over Ω where $p_i = E_P[X_i]$, i.e., the prevision/forecast for X_i is the expected value of X_i under the distribution P. ◊

As a special case, where the variables X_i are indicator functions for events F_i, then the coherent previsions/forecasts are the decision maker's personal probabilities for events, $p_i = P(F_i)$. For simplicity, in the light of this result, we refer to *coherent* quantities when they may be used either as previsions or forecasts.

Example 2: (*A contrast between incoherence$_1$ and incoherence$_2$ when moral hazard is present*.) Suppose that the agent is asked for a pair of *fair* betting odds, one for an event F and one for its complement F^c, and also is asked for forecasts of the same pair of events subject to Brier score. The pair $P(F) = .6$ and $P(F^c) = .9$ are incoherent in both senses, since $P(F) + P(F^c) = 1.5 \neq 1.0$. For demonstrating incoherence$_1$, the opponent chooses $\alpha_F = \alpha_{F^c} = 1$, which produces a sure-loss of .5 for the agent. That is, $(F(\omega) - .6) + (F^c(\omega) - .9) = -.5 < 0$ in each state $\omega \in \Omega$; hence, abstaining from betting, with a constant payoff 0, uniformly dominates the sum of these two *fair* bets in the partition by states Ω. For demonstrating incoherence$_2$, consider the rival coherent forecasts $Q(F) = .35$ and $Q(F^c) = .65$. In states $\omega \in F$, the combined Brier score for the two P-forecasts is $(1-.6)^2 + (0-.9)^2 = .97$ and the combined Brier score for the rival Q-forecasts is $(1-.35)^2 + (0-.65)^2 = .845$. In states $\omega \notin F$, the Brier score for the P-forecasts is $(0-.6)^2 + (1-.9)^2 = .37$ and the Brier score for the rival Q-forecasts is $(0-.35)^2 + (1-.65)^2 = .245$. So, the Brier score for the second pair of forecasts (.35, .65) dominate the Brier score for the first pair (.6, .9) in the partition by states Ω.

If we relax the assumption of act/state independence and permit moral hazard, the two criteria for coherence may diverge. Consider an extreme case of moral hazard. Suppose that conditional on making the incoherent P-forecasts (.6, .9) the agent's conditional probability for event F^c is nearly 1, but conditional on making the rival (coherent) Q-forecasts (.35, .65) the agent's conditional probability for F^c is nearly 0. Then it remains the case that given the incoherent$_1$ pair of betting odds (.6, .9), the agent has a negative conditional expected utility of -0.5 when the opponent chooses $\alpha_F = \alpha_{Fc} = 1$. And offering those incoherent$_1$ betting odds remain dispreferred to abstaining, which has conditional expected utility 0 even in this case of extreme moral hazard. Hence, even with these extreme moral hazards, the robust dominance of abstaining over incoherent$_1$ betting is reflected in the same ranking by conditional expected utility. Even with these extreme moral hazards, the betting argument is unaffected – the Dominance principle, i.e. avoiding a sure-loss, is valid for an agent who maximizes conditional expected utility.

However, with these moral hazards, the conditional expected loss under Brier score given the incoherent$_2$ P-forecast pair (.6, .9) is .37, whereas the conditional expected loss under Brier score given the rival coherent Q-forecast pair (.35, .65) is .845. That is, though the coherent$_2$ forecast pair (.35, .65) dominates the incoherent$_2$ forecast pair (.6, .9) in combined Brier score, the Dominance Principle ranks the options in the wrong preference order when these extreme moral hazards are present. This parallels the situation illustrated in Example 1.

Furthermore, suppose that the same extreme moral hazards apply with all rival Q'-forecasts. That is, suppose that given each rival Q'-forecasts for the pair $\{F, F^c\}$ the agent's conditional probability for event F is 1. Since each rival Q'-forecast pair that dominates (.6, .9) has a combined Brier score greater than .50 when event F obtains, each dominating rival Q'-forecast pair has lower conditional expected utility than the conditional expected utility for the dominated forecasts (.6, .9). In this example, no dominating rival Q'-forecast pair has better conditional expected utility than the incoherent$_2$ pair (.6, .9).◊

In general, when incoherence$_1$ obtains, the agent's most favorable outcome under a sure-loss is dispreferred to the constant 0. The option to abstain robustly dominates the payoffs from a sure-loss. Hence, regardless the moral hazard, the conditional expected utility of a sure-loss is negative, which is less than the conditional expected utility of abstaining, which is 0. Coherence$_1$ is robust against the presence of moral hazard. Example 2 illustrates that coherence$_2$ is not equally robust against moral hazard.

Example 3: (*Combining countably many unconditional previsions when probability is merely finitely additive*.) De Finetti (1949) – (see de Finetti, 1972, p. 91) – noted that when the decision maker's personal probability is merely finitely additive, she/he cannot always accept as *fair* the countable sum of *fair*, coherent$_1$ previsions. That sum may be robustly dominated by abstaining. He argued as follows. Let $\Omega = \{\omega_1, \ldots\}$ be a countable partition. Let W_i be the indicator function for state ω_i: $W_i(\omega) = 1$ if $\omega = \omega_i$ and $W_i(\omega) = 0$ if $\omega \neq \omega_i$. Consider the merely finitely additive coherent$_1$ previsions $E_P[W_i] =$

$p_i \geq 0$ where $\sum_i p_i = c < 1$. So $P(\cdot)$ is not countably additive. Consider the opponent's choice $\alpha_i = -1$, independent of i. Then the utility payoff from combining these infinitely many previsions into a single option is uniformly negative, $\mathbf{G}(\omega, \{p_i\},\{\alpha_i\}) = \sum_i\alpha_i(W_i(\omega) - p_i) = -(1-c) < 0$. Hence, the decision maker's alternative to abstain, with constant payoff 0, uniformly strictly dominates this infinite combination of previsions across Ω.◊

However, if the decision maker's personal probability P is countably additive, then c = 1 and the infinite combination of these coherent$_1$ previsions is not dominated by abstaining, at least for cases where the expectation of the infinite sum of previsions is well defined. So, assume $E_P[\sum_i |\alpha_i(W_i(\omega) - p_i)|] < \infty$. Then countable additivity assures that $E_P[\sum_i\alpha_i(W_i)] = \sum_i E_P[\alpha_i(W_i)]$. So: $E_P[\sum_i\alpha_i(W_i - p_i)] = E_P[\sum_i\alpha_i(W_i)] - \sum_i\alpha_i p_i = \sum_i E_P[\alpha_i(W_i)] - \sum_i\alpha_i p_i = \sum_i\alpha_i p_i - \sum_i\alpha_i p_i = 0$. Therefore, it is not the case that for each $\omega \in \Omega$, $\sum_i\alpha_i(W_i(\omega) - p_i)] < 0$. That is, when probability is countably additive abstaining does not even simply dominate the countable sum of these individually *fair* bets, let alone uniformly dominate the countable sum.

In this section of our paper, we focus on the parallel question whether a coherent$_2$ set of forecasts remain undominated when the Brier scores for countably many forecasts are summed together. As a precaution, if the random variables X_i admit unbounded expectations, $E_P[\sum_i X_i(\omega)] = \infty$, or unbounded second moments, $E_P[\sum_i X_i(\omega)^2] = \infty$, then we face possibly undefined expected utility payoffs when we contrast the Brier-score for the infinite forecast set $\{p_i\}$ with the Brier-score for the rival infinite forecast set $\{q_i\}$.

That is, without further restrictions it may be that $E_P[\sum_i(X_i(\omega) - q_i)^2] = E_P[\sum_i(X_i(\omega) - p_i)^2] = \infty$. Then $E_P[\sum_i(X_i(\omega) - p_i)^2 - \sum_i(X_i(\omega) - q_i)^2] = \infty - \infty$, which is undefined.

In order to avoid this problem, for the remainder of this paper we assume that expectations for sums of these random variables, and for their squares, are *absolutely convergent*:

$$E_P[\sum_i |X_i|] \leq V < \infty \tag{1}$$

$$E_P[\sum_i X_i^2] \leq W < \infty. \tag{2}$$

Of course, a sufficient condition for (1) and (2) obtains when the sums of these random variables are uniformly absolutely convergent

$$\forall \omega \in \Omega,\ \sum_i |X_i(\omega)| \leq T < \infty. \tag{3}$$

However, we develop our argument for the more general case provided under (1) and (2).

The principal difference between dominance for infinite sums of previsions and dominance for infinite sums of Brier scores is expressed by the following result.

***Proposition* 1**: Let P be a finitely additive probability defined over the measure space $\{\Omega, \mathcal{B}\}$ with coherent$_2$ forecasts $E_P[X_i] = p_i$. There does not exist a set of real numbers $\{q_i\}$ such that $\forall \omega \in \Omega,\ \sum_i(p_i - X_i(\omega))^2 - \sum_i(q_i - X_i(\omega))^2 > 0.$◊

(Proofs for the numbered propositions and corollaries are given in the Appendix.)

Proposition 1 asserts that for infinite sums of Brier scores, with forecasts $\{p_i\}$ for $\{X_i\}$ that satisfy (1) and (2), there is no set of rival forecasts $\{q_i\}$ that simply dominate, let alone uniformly strictly dominate the $\{p_i\}$ in Ω. That is, even countably many unconditional coherent$_2$ forecasts cannot be simply dominated under Brier Score.

Example 3 (continued): The following completes our contrast with de Finetti's result about combining countably many *fair* prices for elements of a countable partition. Let $\Omega = \{\omega_i: i = 1, \ldots\}$ be a countable space with P a purely finitely additive probability satisfying $P(\{\omega_i\}) = p_i = 0, i = 1, \ldots$. So $\sum_i p_i = 0 < 1$. As before, let W_i ($i = 1, \ldots$) be the indicator functions for the states in Ω. So. $E_P[W_i] = p_i = 0$ and each wager $\alpha_i(W_i - p_i) = \alpha_i W_i$ is *fair*, i.e. $E_p[\alpha_i(W_i - p_i)] = 0$, for $i = 1, \ldots$. Accepting countably many such previsions and summing them together results in a uniform sure-loss of -1 by setting $\alpha_i = -1$. That is, for each state ω, $\Sigma_i\alpha_i(W_i(\omega) - p_i) = -\Sigma_i W_i(\omega) = -1$, which is a sure loss, as per de Finetti's analysis. However, by *Proposition* 1, there are no rival Q-forecasts $\{q_i\}$ for the $\{W_i\}$ that dominate the P-forecasts $\{p_i = 0\}$ by Brier score in Ω, let alone uniformly dominating these forecasts. Specifically, for each ω, $\Sigma_i|W_i(\omega)| = 1$. So the variables $\{W_i\}$ satisfy (1) and (2), because they satisfy (3). In each state ω the total Brier score for the countably many forecasts $\{p_i = 0\}$ is $\Sigma_i(W_i(\omega) - p_i)^2 = \Sigma_i W_i(\omega)^2 = 1$. *Proposition* 1 asserts that no rival set of forecasts satisfy $(\forall\omega)\ \Sigma_i(W_i(\omega) - q_i)^2 < 1$.◊

Proposition 1, as illustrated by Example 3, shows that the modified decision problem in de Finetti's Prevision game – modified to include infinite sums of betting outcomes – is not isomorphic to the modified forecasting problem under Brier score – modified to include infinite sums of Brier scores. In particular, abstaining from betting, which is the alternative that dominates a Book for coherence$_1$, is not an available alternative under forecasting with Brier score and coherence$_2$.

3. *Dominance for countable sums of conditional previsions/forecasts*

As we reported in Section 1, de Finetti extended both coherence$_1$ and coherence$_2$ to include called-off previsions and called-off forecasts, respectively. In what follows we use $h(\omega)$ as the indicator function for event h.

Definitions:

A called-off prevision p_i for X_i given event h, has utility $\alpha_i h(\omega)(X_i(\omega) - p_i)$ in state ω.

A called-off forecast p_i for X_i given event h, has Brier score, $h(\omega)(X_i(\omega) - p_i)^2$ in state ω.

Note that in case the conditioning event h fails, the called-off prevision and the called-off forecast have a 0 utility outcome. In de Finetti's framework, the two coherence criteria are not applied with infinite sums of called-off previsions or infinite sums of Brier scores from called-off forecasts.

De Finetti (1974) then established that:

(iii) A set of called-off previsions p_i for X_i given h_i are coherent$_1$ if and only if the same quantities p_i are coherent$_2$ called-off forecasts for X_i given h_i, and

(iv) A set of called-off previsions/forecasts given non-null events h_i are coherent if and only there exists a finitely additive conditional probability function $P(\cdot \mid \cdot)$ over $\Omega \times \Omega$ where $p_i = E_P[X_i \mid h_i]$, the P-conditional expected value of X_i given h_i. ◊

Proposition 2, below, shows that a denumerable set of called-off previsions or called-off forecasts may form a dominated option upon combining them into a single payoff. The key consideration is when conditional probabilities are non-conglomerable in a partition.

Let $\pi = \{h_i: i = 1, \dots\}$ be a denumerable partition and let $P(\cdot \mid h_i)$ be the corresponding conditional probability functions associated with the a finitely additive probability P. The conditional probabilities $\{P(\cdot \mid h_i)\}$ are *non-conglomerable* in the partition π provided there exists an event F and $\varepsilon > 0$ where, for each $i = 1, \dots,$

$P(F) \leq P(F \mid h_i) - \varepsilon.$

***Proposition* 2**: When the conditional probabilities $P(F \mid h_i) = p_i$ are not conglomerable in $\pi = \{h_i: i = 1, \dots\}$, then with respect to the partition Ω:

(2.1) abstaining uniformly strictly dominates the countable sum of the individually *fair* called-off previsions, p_i, for F given h_i, and

(2.2) the countable sum of Brier scores from the called-off P-forecasts for F given h_i, p_i, is dominated by the countable sum of Brier scores from a rival set of called-off Q-forecasts.◊

We illustrate *Proposition* 2 with an example of non-conglomerability due to Dubins (1975). This example is illuminating as the conditional probabilities do not involve conditioning on null events.

Example 4: $\Omega = \{\omega_{ij}: i = 1,2; j = 1, \ldots\}$ Let $F = \{\omega_{2j,}\ j = 1, \ldots\}$ and let $h_j = \{\omega_{1j}, \omega_{2j}\}$. Define a merely finitely additive probability P so that $P(\{\omega_{1j}\}) = 0$, $P(\{\omega_{2j}\}) = 2^{-(j+1)}$ for j = 1, … , and let $P(F) = p_F = ½ = P(F^c) = p_{Fc}$. Note that $P(h_j) = 2^{-(j+1)} > 0$, so $P(F \mid h_j) = 1 = p_j$ is well defined by the multiplication rule for conditional probability. Evidently, the conditional probabilities $\{P(F \mid h_j)\}$ are non-conglomerable in π since P(F) = ½ whereas $P(F \mid h_j) = 1$ for j = 1, … .

For (**2.1**), Consider the called off previsions $h_j\alpha_j(F - p_j)$ (j = 1, …) and the unconditional prevision $\alpha_{Fc}(F^c - p_{Fc})$. Choose $\alpha_j = 1 = \alpha_{Fc}$. Then,

$$[\alpha_{Fc}(F^c(\omega) - p_{Fc}) + \Sigma_j h_j(\omega)\alpha_j(F(\omega) - p_j)] = \begin{cases} 0.5 - 1.0 = -0.5, \text{ if } \omega \notin F \\ \\ -0.5 + 0.0 = -0.5, \text{ if } \omega \in F. \end{cases}$$

Hence, these infinitely many individually *fair* previsions are not collectively *fair* when taken together. Their sum is uniformly strictly dominated by 0 in Ω, corresponding to the option to abstain from betting.

Regarding (**2.2**), unlike the situation with *Proposition* 1 involving countably many unconditional forecasts, these called-off forecasts do not satisfy Ω-*UDom* when their scores are combined. That is, the Brier scores for these infinitely many P-forecasts have a combined value:

$$(F^c(\omega) - p_{F^c})^2 + \Sigma_j h_j(\omega)(F(\omega) - p_j)^2 \quad = \quad \begin{cases} 0.25 + 1.00 \ = \ 1.25, \text{ if } \omega \notin F \\ \\ 0.25 + 0.00 \ = \ 0.25, \text{ if } \omega \in F. \end{cases}$$

Consider the rival forecasts $Q(F \mid h_j) = 0.75 = q_j$ and $Q(F^c) = 0.75 = q_{F^c}$. These correspond to the countably additive probability $Q(\{\omega_{1j}\} = 0.25 \times 2^{-j}$ and $Q(\{\omega_{2j}\}) = 0.75 \times 2^{-j}$ for $j = 1, \ldots$. Then the combined Brier score from these countably many rival forecasts is:

$$(F^c(\omega) - q_{F^c})^2 + \Sigma_j h_j(\omega)(F(\omega) - q_j)^2 \quad = \quad \begin{cases} 9/16 \ + 9/16 = \ 1.125, \text{ if } \omega \notin F \\ \\ 1/16 \ + 1/16 = \ 0.125, \text{ if } \omega \in F. \end{cases}$$

Thus, taken together, the score from the countably many non-conglomerable P-forecasts is uniformly, strictly dominated in Ω by the score from the countably many conglomerable Q-forecasts.◊

Last, we establish that when expectations are conglomerable in a partition, then combining the Brier score from countably many called-off forecasts given elements of that partition, or combining the payoffs from countably many called-off previsions given elements of that partition, do not result in a uniform sure loss with respect to Ω.

Definition: Given a finitely additive probability P, its finitely additive expectation $E_P[\bullet]$ is disintegrable in a (countable) partition $\pi = \{h_i: i = 1, \ldots\}$ provided that for each random variable X, $E_P[X] = E_P[\, E_P[X \mid \pi]\,]$. In that case we say that P is disintegrable in π.

Dubins (1975) showed that with respect to the class of bounded random variables, disintegrability of P in a partition π is equivalent to conglomerability of expectations (over all bounded random variables) in that same partition π. In Schervish, Seidenfeld and Kadane (2008) we extend Dubins' result to a linear span of unbounded random variables that have finite first moments and finite conditional first moments.

***Proposition* 3**: Let P be a finitely additive probability disintegrable in $\pi = \{h_i: i = 1, \ldots\}$.

(3.1) The prevision p_{F^c} for F^c together with the countably many called-off previsions p_i for F given h_i, together are not uniformly strictly dominated by 0 in Ω.

(3.2) The Brier score for the unconditional forecast $P(F^c) = p_{F^c}$ summed together with the countably many scores from the called-off forecasts $P(F \mid h_i) = p_i$, $i = 1, \ldots$ is not uniformly strictly dominated in the partition Ω by the combined Brier score from any rival set of forecasts.◊

Propositions 2 and 3 show that when the conditioning events form a countable partition π, coherence$_1$ and coherence$_2$ behave the same when extended to include, respectively, the countable sum of individually coherent called-off previsions, and the total Brier score from called-off forecasts. If and only if these coherent quantities are based on conditional expectations that are conglomerable in π, then no failures of Ω-*UDom* result by combining infinitely many of them. However, each merely finitely probability fails to be conglomerable in some countable partition. Thus, the conjunction of Propositions 1, 2 and 3 identify where the debate whether personal probability may be merely finitely additive runs up against the debate whether to extend either coherence criterion in order to apply it with countable combinations of quantities.

4. *Incentive compatible elicitation of infinitely many forecasts using Brier score.* As we remarked at the end of Section 1, de Finetti's interest in Brier score stemmed from the combination of two of its properties. Coherence$_2$, defined in terms of Brier score, is equivalent to coherence$_1$, defined in terms of fair prices. But, in contrast with fair prices, Brier score provides for incentive compatible elicitation of coherent forecasts, it is a strictly proper scoring rule.

Definitions (i) A scoring rule for coherent forecasts of a random variable X is *proper* for a forecaster whose personal probability is P(·) if the forecaster minimizes expected score by announcing $E_P[X]$.

(ii) A scoring rule for coherent forecasts of a random variable X is *strictly proper* if it is proper and only the quantity $E_P[X]$ minimizes expected score.

Coherence$_1$, based on fair prices, is not proper. Because of the presence of the opponent in the game, who gets to choose whether to buy or to sell the random variable X at the decision maker's announced price R(X), the decision maker faces a strategic choice of pricing. For example, if the decision maker suspects that the opponent's fair price, Q(X), is greater than his own, P(X), then it pays to inflate the announced price and to offer the opponent, e.g., R(X) = [P(X) + Q(X)]/2, rather than offering R(X) = P(X). Thus, the prevision-game as de Finetti defined it for coherence_1 is not incentive compatible for eliciting the decision maker's fair prices.

With a finite set of forecasts, since Brier score is strictly proper for each one, using the finite sum of the Brier scores as the score for the finite set preserves strict propriety. That is, with the sum of Brier scores as the score for the finite set, a coherent forecaster minimizes the expected sum of scores by minimizes each one, and this solution is unique.

Here we report what happens to the propriety of Brier score in each of the three settings of the three *Propositions* presented in Sections 2 and 3. That is, we answer the question whether or not, in each of these three settings, the coherent forecaster minimizes expected score for the infinite sum of Brier scores by announcing her/his coherent prevision for each of the infinitely many variables. These findings are *Corollaries* to the respective *Propositions*.

Corollary 1: Under the same two assumptions, (1) and (2), used to establish *Proposition* 1, the infinite sum of Brier scores applied to the infinite set of forecasts $\{p_i\}$ is a strictly proper scoring rule.

Corollary 2: Under the assumptions used for *Proposition* (2.2), namely when the conditional probabilities $P(F \mid h_i) = p_i$ are not conglomerable in $\pi = \{h_i: i = 1, \ldots\}$, then the infinite sum of Brier scores applied to the infinite set of called-off forecasts $\{p_i\}$ is not proper.

Corollary 3: Under the assumption used to establish *Proposition* (3.2), namely that P is disintegrable in π, the infinite sum of Brier scores applied to the infinite set of called-off P-forecasts $\{p_i\}$ is a proper scoring rule.

Thus, these results about the propriety of infinite sums of Brier scores parallel the respective results about extending coherence$_2$ to allow infinite sums of Brier scores.

5. *Summary and some open questions.* Our focus in this paper is on how two different coherence criteria behave with respect to Dominance when countable sums of random variables are included. *Proposition* 1 shows that, in contrast with *fair* prices for coherence$_1$, when the Brier score from infinitely many unconditional forecasts are summed together there are no new failures of the Dominance Principle for coherence$_2$. That is, if an infinite set of probabilistic forecasts $\{p_i\}$ are even simply dominated by some rival forecast scheme $\{q_i\}$ in total Brier score, then the $\{p_i\}$ are not coherent$_2$, i.e. some finite subset of them is uniformly strictly dominated in Brier score. However, because each merely finitely additive probability fails to be conglomerable in some denumerable partition, in the light of *Proposition* 2, neither of the two coherence criteria discussed here may be relaxed in order to apply the Dominance Principle with infinite combinations of called-off options. That change would restrict coherent called-off previsions and called-off forecasts to the set of conditional expectations from countably additive probabilities. Merely finitely additive probabilities then would become *incoherent*.

Specifically, the conjunction of *Propositions* 1, 2 and 3 shows it matters only in cases that involve non-conglomerability that incoherence$_2$ is established using Brier score from a *finite* than from an *infinite* combination of forecasts. In that one aspect, we think coherence$_2$ constitutes an improved version of the concept of *coherence*. Coherence$_1$ applied to a merely finitely additive probability leads to failures of the Dominance Principle both with infinite combinations of unconditional and infinite combinations of non-conglomerable conditional probabilities. Coherence$_2$ leads to failures of the Dominance Principle only with infinite combinations of non-conglomerable conditional probabilities. However, regarding other concerns, e.g., when moral hazards are introduced, the Dominance Principle applies to coherence$_1$ though not to coherence$_2$.

The results reported here contrast coherence$_1$ and coherence$_2$. Coherence$_2$ is formulated in terms of Brier score. However, as there exists a continuum of different (strictly) proper scoring rules, the question naturally arises whether the three numbered propositions of this paper generalize to the other versions of coherence that might be formulated using a different proper scoring rule other than Brier score. In Schervish, Seidenfeld and Kadane (2009) we establish that all these different versions of coherence are equivalent in their usual formulation. That is, a finite set of (called-off) forecasts admits a uniformly strictly dominating rival set of (called-off) forecasts under any one proper scoring rule just in case it does under another. We conclude with a conjecture that, therefore, the results shown here for Brier scores also hold for a much wider class of scoring rules.

Conjecture: With suitable restatement of assumptions (1) and (2) in order to match the particular scoring rule, *Propositions* 1-3 generalize to all strictly proper scoring rules.

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Appendix

Recall the two assumptions:

$$E_P[\ \sum_i|X_i|\ ] \ \leq\ V\ <\ \infty. \qquad (1)$$

$$E_P[\ \sum_i X_i^2\ ] \ \leq\ W\ <\ \infty. \qquad (2)$$

***Proposition* 1**: There does not exist a set of real numbers $\{q_i\}$ such that

$$\forall\omega \in \Omega,\ \sum_i(p_i - X_i(\omega))^2 - \sum_i(q_i - X_i(\omega))^2\ >\ 0. \qquad (4)$$

Proof: To establish *Proposition* 1 it is sufficient, provided that some $q_i \neq p_i$, for

$$E_P[\sum_i(q_i - X_i(\omega))^2 - \sum_i(p_i - X_i(\omega))^2]\ >\ 0. \qquad (5)$$

Because of (1) and (2),

$$E_P[\sum_i(q_i - X_i)^2 - \sum_i(p_i - X_i)^2]\ = E_P[\sum_i ([q_i - X_i]^2 - [p_i - X_i]^2)]. \qquad (6)$$

Next, observe that with respect to finite sums:

$$E_P[\sum_{i=1}^{k}\ [(q_i - X_i(\omega))^2 - (p_i - X_i(\omega))^2]\ = \sum_{i=1}^{k}(q_i - p_i)^2 \geq 0, \qquad (7)$$

And if for at least one value of $i \leq k$, $q_i \neq p_i$,

$$\sum_{i=1}^{k}(q_i - p_i)^2 > 0. \qquad (8)$$

Define the random variables:

$$X_i^+(\omega) = \ \mathit{max}\{0, X_i(\omega)\}$$

$$X_i^-(\omega) = \mathit{min}\{0, X_i(\omega)\}$$

$$Z_i(\omega) = |X_i(\omega)|$$

$$X(\omega) = \sum_i X_i(\omega)$$

and $$Z(\omega) = \sum_i Z_i(\omega).$$

Then $p_i = E_P(X_i) = E_P[X_i^-] + E_P[X_i^+]$,

and let $r_i = E_P(Z_i) = |E_P[X_i^-]| + E_P[X_i^+]$.

So $|p_i| \le r_i$. By (1), $E_P[Z] \le V$. Since P is finitely additive, $\sum_i|p_i| \le \sum_i r_i \le E_P[Z] \le V$. Hence $\{p_i\}$, and therefore also $\{p_i^2\}$, are absolutely convergent. From this we argue that:

$$E_P[\sum_i|(p_i - X_i)|] < \infty, \text{ and} \tag{9}$$

$$E_P[\sum_i(p_i - X_i)^2] < \infty. \tag{10}$$

Note that $E_P[\sum_i |(p_i - X_i)|] \le \sum_i|p_i| + E_P[Z]$. Therefore $E_P[\sum_i |(p_i - X_i)|] \le 2V$, which is sufficient for (9).

In order to establish (10) note that

$E_P[\sum_i(p_i - X_i)^2] = \sum_i p_i^2 - 2E_P[\sum_i p_i X_i] + E_P[\sum_i X_i^2]$, which we write as (A) + (B) + (C).

(A) $\sum_i p_i^2 < \infty$, as shown above.

(B) $\forall\omega \in \Omega \ \ |\sum_i p_i X_i(\omega)| \le \sum_i|p_i X_i(\omega)| \le \mathit{sup}\{|p_i|\}Z(\omega) \le \mathit{sup}\{|p_i|\}V < \infty$.

Therefore, $E_P[|\sum_i p_i X_i|] \le \mathit{sup}\{|p_i|\}V < \infty$.

(C) By (2), $E_P[\sum_i X_i^2] \le W < \infty$.

Next, in the light of (7), write $E_P[\sum_i [(q_i - X_i)^2 - (p_i - X_i)^2]] =$

$$E_P[\sum_{i=1}^{k} [(q_i - X_i)^2 - (p_i - X_i)^2]] + E_P[\sum_{i=k+1}^{\infty} [(q_i - X_i)^2 - (p_i - X_i)^2]] = \qquad (11)$$

$$\sum_{i=1}^{k} (q_i - p_i)^2 + E_P[\sum_{i=k+1}^{\infty} [(q_i - X_i)^2 - (p_i - X_i)^2]] . \qquad (12)$$

Case 1: $\sum_i (q_i - p_i)^2 = \infty$.

Choose k sufficiently large so that $\sum_{i=1}^{k} (q_i - p_i)^2 > E_P[\sum_{i=k+1}^{\infty} (p_i - X_i)^2]$, which is easy to do since, by (10), $E_P[\sum_i(p_i - X_i)^2]$ is finite. Thus equation (5) follows.

Case 2: $\sum_i(q_i - p_i)^2 = T$, with $0 < T < \infty$.

Note that $E_P[\sum_i((q_i - X_i)^2 - (p_i - X_i)^2)] =$

$$\sum_{i=1}^{k} (q_i - p_i)^2 + E_P[\sum_{i=k+1}^{\infty} ((q_i - p_i)^2 + 2(q_i - p_i)(p_i - X_i))] = \qquad (13)$$

$$\sum_i(q_i - p_i)^2 + E_P[\sum_{i=k+1}^{\infty} 2(q_i - p_i)(p_i - X_i)] = \qquad (14)$$

$$T + E_P[\sum_{i=k+1}^{\infty} 2(q_i - p_i)(p_i - X_i)] \qquad (15)$$

Choose k sufficiently large so that $T > |E_P[\sum_{i=k+1}^{\infty} 2(q_i - p_i)(p_i - X_i)]|$. That is, given (9), let $0 \leq E_P[\sum_i|(p_i - X_i)|] = U < \infty$. Since $\sum_i(q_i - p_i)^2 = T < \infty$, $lim_i(q_i - p_i) = 0$. Then choose k sufficiently large so that for each $i > k$, $|(q_i - p_i)|U < T/3$. Again, equation (5) follows.◊

Corollary 1: Under the same two assumptions, (1) and (2), used to establish *Proposition* 1, the infinite sum of Brier scores applied to the infinite set of forecasts $\{p_i\}$ is a strictly proper scoring rule.

Proof: The corollary is equivalent to the claim that, provided some $q_i \neq p_i$,

$$E_P[\sum_i(q_i - X_i(\omega))^2 - \sum_i(p_i - X_i(\omega))^2] > 0.$$

This is equation (5), which is established in the proof of *Proposition* 1.◊

***Proposition* 2**: When the conditional probabilities $P(F \mid h_i) = p_i$ are not conglomerable in $\pi = \{h_i: i = 1, \ldots\}$, then with respect to the partition Ω:

(2.1) abstaining uniformly strictly dominates the countable sum of the individually *fair* called-off previsions, p_i, for F given h_i, and

(2.2) the countable sum of Brier scores from the called-off forecasts for F given h_i, p_i, is dominated by the countable sum of Brier scores from a rival set of called-off forecasts.◊

Proof: Let $\pi = \{h_i: i = 1, \ldots\}$ be a denumerable partition. Assume that for some event F and $\varepsilon > 0$, $P(F) \leq P(F \mid h_i) - \varepsilon$, so that these conditional probabilities are not conglomerable. Write the unconditional probability $P(F) = p_F$, and the countably many conditional probabilities, $P(F \mid h_i) = p_i$, $i = 1, \ldots$. Without loss of generality, let $\varepsilon = inf_i\{p_i\} - P(F) > 0$.

(**2.1**) Based on this finitely additive probability P, consider the unconditional prevision $P(F^c) = 1-p_F$, with payoff $\alpha(F^c(\omega) - (1-p_F))$ and the countably many called-off previsions for F, given h_i, with payoffs $\alpha_i h_i(\omega)(F(\omega) - p_i)$. For each $\omega \in \Omega$, let h_{i_ω} be the unique element of the partition π such that $\omega \in h_{i_\omega}$, i.e., $h_j(\omega) = 1$ if and only if $j = i_\omega$. Since, for each i, $(1-p_F) + p_i > 1+\varepsilon$, let the opponent choose $\alpha = \alpha_i = 1$. Then, for each ω,

$$(F^c(\omega) - (1\text{-}p_F)) + \Sigma_i h_i(\omega)(F(\omega) - p_i) =$$

$$(F^c(\omega) - (1\text{-}p_F)) + (F(\omega) - p_{i_\omega}) = 1 - [(1\text{-}p_F)+ p_{i_\omega}] \leq -\varepsilon < 0.$$

Thus the countable sum of the called-off previsions for F given h_i, combined with the prevision for F^c results in a payoff that is uniformly strictly dominated by 0.

(**2.2**) Next we show that the parallel result holds also when these conditional probabilities are used as called-off forecasts. Adding the Brier scores for the unconditional forecast of F^c together with the countably many called-off forecasts for F given h_i yields, in state ω, the total score:

$$(F^c(\omega) - (1\text{-}p_F))^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2 =$$

$$(F^c(\omega) - (1\text{-}p_F))^2 + h_{i_\omega}(\omega)(F(\omega) - p_{i_\omega})^2 = \begin{cases} p_F^{\,2} + p_{i_\omega}^{\,2} & \text{if } \omega \notin F \\ (1\text{-}p_F)^2 + (1\text{-}p_{i_\omega})^2 & \text{if } \omega \in F. \end{cases} \tag{16}$$

Consider a set of rival forecasts $Q(F^c) = q_{Fc}$ and $Q(F \mid h_i) = q_i$ ($i = 1, \ldots$). The total Brier score for a set of rival Q-forecasts in state ω is:

$$(F^c(\omega) - q_{Fc})^2 + \Sigma_i h_i(\omega)(F(\omega) - q_i)^2 = (F^c(\omega) - q_{Fc})^2 + h_{i_\omega}(\omega)(F(\omega) - q_{i_\omega})^2. \tag{17}$$

We identify dominating rival Q-forecasts. That is, we give rival Q-forecasts so that (17) is uniformly smaller than (16) for each state $\omega \in \Omega$.

Let $p_L = \mathit{inf}_{h\in\pi}\{P(F|h)\} = \mathit{inf}_i\{p_i\}$.

Define $q_{Fc} = 1\text{-} (p_F+p_L)/2 = 1 - (p_F + \varepsilon/2)$ and $q_i = p_i - \varepsilon/2$.

As given by (17), in state ω, the total Brier score for these rival Q-forecasts is:

If $\omega \notin F$, $(F^c(\omega) - q_{F^c})^2 + h_{i_\omega}(\omega)(F(\omega) - q_{i_\omega})^2$

$$= (1-q_{F^c})^2 + q_{i_\omega}{}^2 = (p_F + \varepsilon/2)^2 + (p_{i_\omega} - \varepsilon/2)^2$$

$$= p_F{}^2 + p_{i_\omega}{}^2 - \varepsilon(p_{i_\omega} - (p_F + \varepsilon/2))$$

$$\leq p_F{}^2 + p_{i_\omega}{}^2 - \varepsilon^2/2$$

$$< p_F{}^2 + p_{i_\omega}{}^2$$

$$= (F^c(\omega) - (1-p_F))^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2$$

And if $\omega \in F$, $(F^c(\omega) - q_{F^c})^2 + h_{i_\omega}(\omega)(F(\omega) - q_{i_\omega})^2$

$$= q_{F^c}{}^2 + (1-q_{i_\omega})^2 = (1-(p_F + \varepsilon/2))^2 + (1-(p_{i_\omega} - \varepsilon/2))^2$$

$$= (1-p_F)^2 + (1-p_{i_\omega})^2 - \varepsilon[p_{i_\omega} - p_F - \varepsilon/2]$$

$$\leq (1-p_F)^2 + (1-p_{i_\omega})^2 - \varepsilon^2/2$$

$$< (1-p_F)^2 + (1-p_{i_\omega})^2$$

$$= (F^c(\omega) - (1-p_F))^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2.$$

Thus, for each state ω, (17) is less than (16) by at least $\varepsilon^2/2$. Hence, these rival Q-forecasts uniformly dominate the P-forecasts in total Brier score.◊

The dominating Q-forecasts admit a simple geometric interpretation based on the elementary fact that the total Brier score is given by the square of an l_2-norm. For each i, the unconditional P-probability for F^c plus the P-conditional probability for F given h_i sum to at least $1 + \varepsilon$, i.e., $[(p_{F^c} + p_i] \geq 1 + \varepsilon > 1$. As depicted in Figure 1, below, represent each pair, (p_{F^c}, p_i), as point $\mathbf{p}^\mathbf{i}$ in the unit square. The lower right corner of the square (1,0) is identified with any state ω in which F fails to occur, so then $F^c(\omega) = 1$ and $F(\omega) = 0$. The diagonally opposite upper left corner, (0,1), is identified with any state in which event F obtains. Each such point $\mathbf{p^i}$ = (p_{Fc}, p_i) from the P-forecasts lies on a vertical line segment $\overline{\mathrm{P}}$ that sits above the reverse diagonal ***D*** connecting the two corners, (0,1) and (1,0). Denote by $\mathbf{p^L}$ the point with coordinates $((1\text{-}p_F), p_L)$. As depicted in Figure 1, the lower endpoint of $\overline{\mathrm{P}}$ is given by $\mathbf{p^L}$. Without loss of generality, we may choose the upper endpoint of $\overline{\mathrm{P}}$ to be $(1\text{-}p_F,1)$. Last, denote by $\mathbf{r^i}$ the mirror image of point $\mathbf{p^i}$ reflected across ***D***.

Each rival Q-forecast that has a better total Brier score than does P at ω, i.e. the rival forecasts where (17) is smaller than (16), are given by points that sit inside a circle centered at the respective corner, depending upon whether $F(\omega) = 0$ or $F(\omega) = 1$, and whose radius equals the square-root of the Brier score for the combined P-forecasts in that state. If $F(\omega) = 1$, that circle is centered at the point (0,1) and has radius $\sqrt{[p_{Fc}^2 + (1 - p_{i\omega})^2]}$. If $F(\omega) = 0$, that circle is centered at the point (1,0) and has radius $\sqrt{[p_F^2 + p_{i\omega}^2]}$. The circumference of each circle passes through the point $\mathbf{p}^{\mathbf{i}\omega} = ([1\text{-}p_F], p_{i\omega})$.

For each $h_i \in \pi$, either one or two such circles exist. Specifically, the point $\mathbf{p^i}$ belong to two such circles if and only if both $h_i \cap F \neq \phi$ and $h_i \cap F^c \neq \phi$ and at least one of these two inequalities obtains since $h_i \neq \phi$. In order for (17) to be less than (16) for each ω, i.e., in order for the countably many rival Q-forecasts to dominate the countably many P-forecasts, it is necessary and sufficient to find points $\mathbf{q}^i = (q_{Fc}, q_i)$ that fall within each of the (one or) two circles for which there is a corresponding $\omega \in h_i$. The geometric argument for *Proposition* 2 is completed by explaining the construction depicted in Figure 1 for finding these uniformly dominating rival Q-forecasts. They fall on line segment $\overline{Q}$.

Identify the point $\mathbf{q^L}$ as the projection of point $\mathbf{p^L}$ onto the reverse diagonal ***D***. The coordinates of $\mathbf{q^L}$ are (q_{Fc}, (1-q_{Fc})) where q_{Fc} = 1- (p_F+p_L)/2 = 1 - (p_F + ε/2). Let $\overline{Q}$ be the vertical line segment whose lower end point is $\mathbf{q^L}$ and whose upper end point is (q_{Fc},1). The rival Q-forecasts are found by mapping a point $\mathbf{p^i}$ from $\overline{P}$ to the point $\mathbf{q^i}$, where $\mathbf{q^i}$ is the intersection of the line projecting $\mathbf{p^i}$ onto ***D*** and the vertical line segment $\overline{Q}$. That is the point $\mathbf{q^i}$ = (q_{Fc}, q_i), where q_i = p_i - ε/2, is a dominator for point $\mathbf{p^i}$.

For each ω ∈ h_i, the point $\mathbf{q^i}$ falls within the respective lens of points that have a better Brier score than does the point $\mathbf{p^i}$. That is, $\mathbf{q^i}$ falls on the interior of the line segment connecting the two points $\mathbf{p^i}$ and $\mathbf{r^i}$. That line segment is a chord in the (one or) two circles that share points $\mathbf{p^i}$ and $\mathbf{r^i}$ on their circumferences and which identify the set of rival forecasts that dominate the forecasts associated with point $\mathbf{p^i}$.

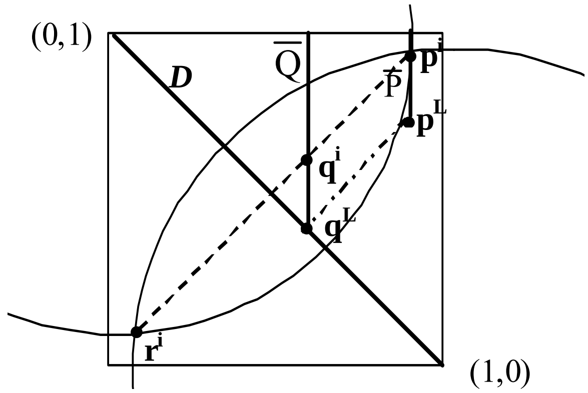

Figure 1

Corollary 2: Under the assumptions used for *Proposition* (2.2), namely when the conditional probabilities $P(F \mid h_i) = p_i$ are not conglomerable in $\pi = \{h_i: i = 1, \ldots\}$, then the infinite sum of Brier scores applied to the infinite set of called-off forecasts $\{p_i\}$ is not proper.

Proof: The corollary is immediate from *Proposition* (2.2), as the existence of the rival set of dominating Q-forecasts, $\{q_i\}$, establishes that the forecaster does not minimize her/his expected Brier score for the called-off quantities $\{h_iF\}$ by giving their called-off previsions $\{p_i\}$.◊

***Proposition* 3** Let P be a finitely additive probability disintegrable in $\pi = \{h_i: i = 1, \ldots\}$.

(3.1) The prevision p_{F^c} for F^c together with the countably many called-off previsions p_i for F given h_i, together are not uniformly strictly dominated by 0 in Ω.

(3.2) The unconditional forecast $P(F^c) = p_{F^c}$ together with the countably many called-off forecasts $P(F \mid h_i) = p_i$, $i = 1, \ldots$ are not uniformly strictly dominated in Brier score in the partition Ω. That is, in state ω the total Brier score for the P-forecasts, taken together is

$$(F^c(\omega) - p_{F^c})^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2.$$

Then there is no rival set of Q-forecasts $\{q_{F^c}, q_i: i = 1, \ldots\}$ whose Brier score uniformly dominates in Ω. That is, there is no rival set of forecasts such that for some $\varepsilon > 0$ and every ω

$$(F^c(\omega) - p_{F^c})^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2 - [(F^c(\omega) - q_{F^c})^2 + \Sigma_i h_i(\omega)(F(\omega) - q_i)^2] > \varepsilon. \diamond$$

Proof:

(**3.1**) For event F consider the unconditional prevision $P(F^c) = p_{F^c}$ and the countably many conditional forecasts $P(F \mid h_i) = p_i$, $i = 1, \ldots$.

Consider the net payoff in state ω, given by the sum

$$\alpha(F^c(\omega) - p_{F^c}) + \Sigma_i \alpha_i h_i(\omega)(F(\omega) - p_i). \tag{18}$$

In order to show that (18) cannot be uniformly strictly negative, it is sufficient to show

$$E_P[\alpha(F^c - p_{F^c}) + \Sigma_i \alpha_i h_i(F - p_i)] = 0. \tag{19}$$

Of course, $E_P[\alpha(F^c - p_{F^c}) + \Sigma_i \alpha_i h_i(F - p_i)] = E_P[\alpha(F^c - p_{F^c})] + E_P[\Sigma_i \alpha_i h_i(F - p_i)]$.

Trivially,

$$E_P[\alpha(F^c - p_{F^c})] = 0 \tag{20}$$

Since P is disintegrable in π,

$$E_P[\Sigma_i \alpha_i h_i(F - p_i)] = E_P[\, E_P[\Sigma_i \alpha_i h_i(F - p_i)] \mid \pi] \,]. \tag{21}$$

However, given an $h_j \in \pi$, $P(\mathit{max}_{i \neq j}\{h_i\} = 0 \mid h_j) = 1$ for $i \neq j$, and of course $P(h_j = 1 \mid h_j) = 1$.

Hence,

$$E_P[\Sigma_i \alpha_i h_i(F - p_i)] \mid h_j] = E_P[[\alpha_j h_j(F - p_j)] \mid h_j\,] \tag{22}$$

and trivially, $E_P[[\alpha_j h_j(F - p_j)] \mid h_j\,] = 0$

Thus

$$E_P[\Sigma_i \alpha_i h_i(F - p_i)] = 0. \tag{23}$$

Equations (20) and (23) establish (19).

(3.2) In state ω the total Brier score for the countably many P-forecasts is

$$(F^c(\omega) - p_{F^c})^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2 \tag{24}$$

Then for (3.2) we must establish that there is no rival set of forecasts $\{q_{F^c}, q_i: i = 1, \ldots\}$ whose Brier score uniformly dominates (24). That is, there is no rival set of forecasts such that for some $\varepsilon > 0$ and every ω

$$(F^c(\omega) - p_{F^c})^2 + \Sigma_i h_i(\omega)(F(\omega) - p_i)^2 - [(F^c(\omega) - q_{F^c})^2 + \Sigma_i h_i(\omega)(F(\omega) - q_i)^2] > \varepsilon. \tag{25}$$

It is sufficient to show that

$$E_P[(F^c - q_{F^c})^2 + \Sigma_i h_i(F - q_i)^2 - [(F^c - p_{F^c})^2 + \Sigma_i h_i(F - p_i)^2]\,] \geq 0. \tag{26}$$

Write the left-hand side of (26) as

$$E_P[(F^c - q_{F^c})^2 - (F^c - p_{F^c})^2] + E_P[\Sigma_i h_i(F - q_i)^2 - \Sigma_i h_i(F - p_i)^2]. \tag{27}$$

Observe that

$$E_P[(F^c - q_{F^c})^2 - (F^c - p_{F^c})^2] = (q_{F^c} - p_{F^c})^2 \geq 0. \tag{28}$$

From the assumption that P is disintegrable in π,

$$E_P[\Sigma_i h_i(F - q_i)^2 - \Sigma_i h_i(F - p_i)^2] = E_P[E_P[\Sigma_i h_i(F - q_i)^2 - \Sigma_i h_i(F - p_i)^2 \mid \pi]]. \tag{29}$$

Using the same reasoning as above, we obtain,

$$E_P[\Sigma_i h_i(F - q_i)^2 - \Sigma_i h_i(F - p_i)^2 \mid h_j] = E_P[h_j[(F - q_j)^2 - (F - p_j)^2] \mid h_j] = (q_j - p_j)^2 \geq 0. \tag{30}$$

Therefore, since P is disintegrable in π,

$$E_P[\Sigma_i h_i(F - q_i)^2 - \Sigma_i h_i(F - p_i)^2] \geq 0. \tag{31}$$

Equations (28) and (31) establish (27).◊

Corollary 3: Under the assumption used to establish *Proposition* (3.2), namely that P is disintegrable in π, the infinite sum of Brier scores applied to the infinite set of called-off P-forecasts $\{p_i\}$ is a proper scoring rule.

Proof: The corollary is equivalent to the claim that for each set of rival called-off Q-forecasts, $\{q_i\}$,

$$E_P[\ \Sigma_i h_i(F - q_i)^2 - \Sigma_i h_i(F - p_i)^2\ ] \ \geq\ 0.$$

This is equation (31), which is established in the proof of *Proposition* (3,2).◊