Quantiles for Complete Lattices

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Motivation

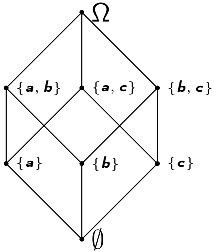
- Analysis of set-valued estimators
- Construction of confidence sets for set-valued estimators
- More general applications thinkable

Order theory

Definition

*A **partially ordered set** (poset) $\mathbb{A} = (A, \leq)$ is a set A with a relation $\leq$ that is reflexive, transitive and antisymmetric. A poset is called **lattice** if every two elements have a least upper bound (supremum, join) and a greatest lower bound (infimum, meet). It is called **complete lattice** if every arbitrary set has a join and a meet.*

Example:



Definition

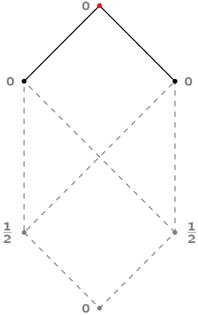
Let $(\mathbb{A}, \mathcal{F}, m)$ be a probability space where $\mathbb{A}$ is a partially ordered set. If $\mathcal{F}$ contains all principal ideals $\downarrow x$ then the corresponding belief function $B_m : \mathbb{A} \longrightarrow [0,1]$ is defined as

$$B_m(x) := m(\downarrow x) = m(\{y \in \mathbb{A} \mid y \leq x\}).$$

An element $a \in \mathbb{A}$ is called an α-**quantile** (for m) if $B_m(a) = \alpha$ and a is minimal in $B_m^{-1}(\alpha)$. It is called **quantile** (for m) if it is an α-quantile for some α. If we have only $B_m(a) = \alpha$ we say that a is an α-**prequantile**.

Minimality?

$B_m^{-1}(\alpha):$



Minimality?

Lemma

Let $(\mathbb{A}, \mathcal{F}, m)$ *be a probability space (for which all principal ideals* $\downarrow x$ *are measurable). The set*

$$\mathfrak{Q} := \{a \in \mathbb{A} \mid \ a \text{ is minimal in } B_m^{-1}(B_m(a))\}$$

of all quantiles is a kernel system. (A kernel system is a nonempty system that is closed under arbitrary joins and contains the smallest element $\perp$*.)*

Remark

More generally for a monotone and supermodular mapping $B : \mathbb{A} \longrightarrow \mathbb{M}$ in a partially ordered quasi cancellative monoid $\mathbb{M}$ the system

$$\mathfrak{Q} := \{a \in \mathbb{A} \mid \ \text{a is minimal in } B^{-1}(B(a))\}$$

is a kernel system.

Minimality?

Lemma

Let $(\mathbb{A}, \mathcal{F}, m)$ *be a probability space (for which all principal ideals* $\downarrow x$ *are measurable). The set*

$$\mathfrak{Q} := \{a \in \mathbb{A} \mid \ a \text{ is minimal in } B_m^{-1}(B_m(a))\}$$

of all quantiles is a kernel system. (A kernel system is a nonempty system that is closed under arbitrary joins and contains the smallest element $\bot$*.)*

Corollary

Let a *be an* α*-prequantile. Then*

$$k_{B_m}(a) := \bigvee_{q \in \mathfrak{Q}, q \leq a} q$$

is a quantile.

Question: $\quad B_m(k_{B_m}(a)) = \alpha$?

Definition

A complete lattice $\mathbb{A}$ is called ***linearly order colindelöf*** *if every chain C contains an at most countable subchain S with*

$$\bigwedge S = \bigwedge C.$$

Minimality?

Lemma

Let a be an α-prequantile. Then

$$k_{B_{\boldsymbol{m}}}(a) := \bigvee_{q \in \mathfrak{Q}, q \leq a} q$$

is a quantile. If $\mathbb{A}$ is linearly order colindelöf then

$$B(k_{B_{\boldsymbol{m}}}(a)) = \alpha.$$

Remark

This also works with a monotone and supermodular function $B : \mathbb{A} \longrightarrow [0, 1]$ induced by a (not necessarily nonnegative) möbius inverse that is continuous from above.

Example

$\mathbb{A} = (2^{[0,1]}, \subseteq)$ is not linearly order colindelöf:

- Take the set of all cocountable subsets of $[0, 1]$ and choose a maximal chain in this set. (This is possible because of Zorn's lemma.)
- Then $\bigwedge T = \emptyset$ because if there was an element $x \in \bigwedge T$, the chain $T \cup \{\bigwedge T \setminus \{x\}\}$ would be a strict superchain of T which contradicts the maximality of T.
- For a countable subchain S we have $(\bigwedge S)^c = \bigvee \{s^c \mid s \in S\}$ which is a countable union of countable sets, thus countable.
- So $\bigwedge S$ must be uncountable and thus nonempty.
- This shows that there does not exist any countable subchain S with $\bigwedge S = \bigwedge T$.

Example

Furthermore there exists a probability measure m on

$$\mathcal{F} := \{S \subseteq \mathcal{B}([0,1]) \mid \{a \in S \mid \quad |a| = 1\} \in \mathcal{B}([0,1])\}$$

such that $B_m^{-1}(1)$ has no minimal elements. Take for example $m : \mathcal{F} \longrightarrow [0,1] : A \mapsto \lambda(\{x \in [0,1] \mid \{x\} \in A\})$ with λ the Lebesgue measure. It is clear that every set $X \in B_m^{-1}(1)$ has uncountably many elements and for an arbitrary element $x \in X$ we have $X \backslash \{x\} \in B_m^{-1}(1)$ which means that there cannot be minimal elements in $B_m^{-1}(1)$.

Representation invariance

Definition

A mapping $f : \mathscr{P}(\mathbb{A}) \longrightarrow \mathbb{A}$ is called **representation invariant** if for every (bimeasurable) order automorphism Ψ on $\mathbb{A}$ and every map $g \in \mathscr{P}(\mathbb{A})$ we have

$$f(g) = \Psi(f(g \circ [\Psi]))$$

with

$$[\Psi] : \mathcal{F} \longrightarrow \mathcal{F} : X \mapsto \Psi[X] := \{\Psi(x) \mid x \in X\}.$$

Lemma

Let $f : \mathscr{P}(\mathbb{A}) \longrightarrow \mathbb{A}$ be a representation invariant map. Then the mapping

$$f_k : \mathscr{P}(\mathbb{A}) \longrightarrow \mathbb{A} : m \mapsto k_{B_{\boldsymbol{m}}}(f(m)) = \bigvee\{q \mid q \in \mathcal{K}_{B_{\boldsymbol{m}}}, q \leq f(m)\}$$

is a representation invariant, quantile-valued mapping.

Weak quantiles

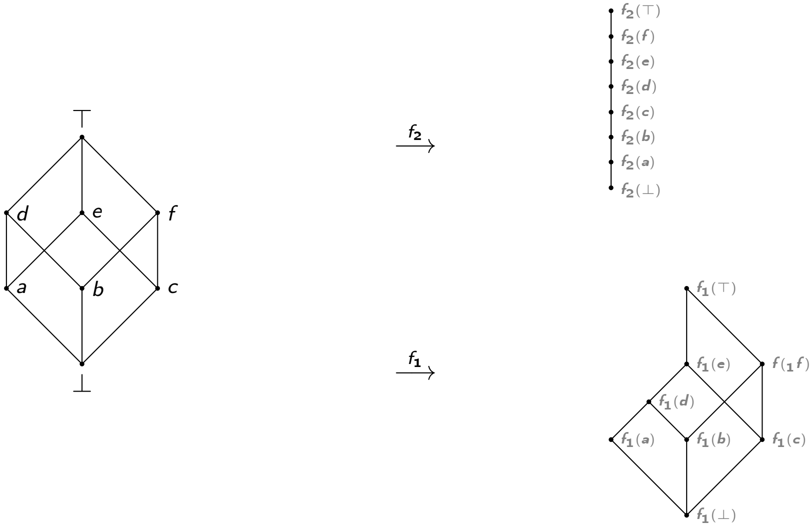
Definition

Let $m : \mathcal{F} \longrightarrow [0,1]$ be a probability measure (where all principal ideals are measurable) and let $\mathcal{C}$ be a class of functions with domain $\mathbb{A}$ and partially ordered codomains. We say that $a \in \mathbb{A}$ is a weak α quantile (for m) if it is a minimal element of the set

$$\{x \in \mathbb{A} \mid \forall f \in \mathcal{C} : m(f^{-1}(\downarrow (f(x)))) \geq \alpha \ \ \& \ \ \exists g \in \mathcal{C} : m(g^{-1}(\downarrow (g(x)))) = \alpha\}.$$

Analogously a is called **weak quantile** (with respect to $\mathcal{C}$ and m) if it is a weak α-quantile for some α.

Weak quantiles



What about representation invariant weak quantile mappings?