A note on sharp identification regions

Definition

Let $P := \{\mathbb{P}_\theta \mid \theta \in \Theta\}$ *be a statsitical model and*

- $Y, \ldots$ *unobservable random variables,*
- $X, \underline{Y}, \overline{Y}, \ldots$ *observable random variables w.r.t an underlying probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*
- *The joint distribution of the random Variables* $X, Y, \underline{Y}, \overline{Y}$ *under a model* P_θ *is denoted with* F_θ *and the joint distribution under the „true model"* $\mathbb{P}$ *is denoted with* $F^{X,Y,\underline{Y},\overline{Y}}$*.*
- *The unobserved variables fullfill a certain condition* $C(X, Y, \underline{Y}, \overline{Y}) = 1$*.*
  e.g. $\underline{Y} \leq Y \leq \overline{Y}$ *or* $\forall X : \mathbb{E}(\underline{Y} \mid X) \leq \mathbb{E}(Y \mid X) \leq \mathbb{E}(\overline{Y} \mid X)$*.*

Definition

- *Two parameters θ_1 and θ_2 are undistinguishable (i.e. $\theta_1 \sim \theta_2$) if the corresponding models $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$ are empirically undistinguishable, which means, that the distributions of the observable variables are the same:*

$$F_{\theta_1}^{X,\underline{Y},\overline{Y}} = F_{\theta_2}^{X,\underline{Y},\overline{Y}}.$$

Definition

A statistical model P is called point-identified, if any two different parameters θ_1 *and* θ_2 *are empirically distinguishable, i.e.:*

$$\sim \quad = \quad \Delta_\Theta = \{(\theta, \theta) \mid \theta \in \Theta\}.$$

Otherwise it is called partially identified.

Example

The simple linear model

$$\Theta = B \times \mathbb{R}_{\geq 0} \times \mathcal{Z}(\mathbb{R}_{\geq 0}) \times \mathcal{Z}(\mathbb{R}_{\geq 0})$$

with $B = \mathbb{R}^2$. For $\theta = (\beta, \sigma^2, \sigma_l, \sigma_u) \in \Theta$, the random variables are defined as:

$$\begin{aligned} Y &= X\beta + \varepsilon \\ \underline{Y} &= X\beta + \varepsilon - \sigma_l \\ \overline{Y} &= X\beta + \varepsilon + \sigma_u \end{aligned}$$

with $\varepsilon \sim N(0, \sigma^2 I)$.

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with $\varepsilon \sim N(0, \sigma^2 I)$.

Here we are only interested in the values of $\beta \in B$.

This model is only partially identified. For example

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Moreover, the quotient space $\Theta_{/\sim}$ *ist not of the form*

$$\Theta_{/\sim} \quad = \quad B_{/\approx} \times \text{„rest"},$$

so we must factorize the whole space Θ *and not only the interesting* B *to make the model point-identified.*

Estimation $\longleftarrow$ *Model* $\longrightarrow$ *Pediction*

Estimation ⟵ *Model* ⟶ *Pediction*

„model as a truth to be estimated "

„model as a tool to be applied"

Estimation ⟵ Model ⟶ Pediction

„model as a truth to be estimated “ „model as a tool to be applied“

e.g.: least squares estimator ⟵ linear model ⟶ best linear predictor

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\iff \quad & F^Y = F^Y_\theta
\end{aligned}
$$

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$$
\begin{aligned}
& Y \sim F_\theta \\
\iff \quad & F^Y = F_\theta^Y \\
\iff \quad & L(F_\theta^Y, F) = 0
\end{aligned}
$$

for some distance-function $L(\cdot, \cdot)$*.*

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- *if the model is correctly specified, then „prediction“ and „estimation“ are „nearly the same“.*

The actual problem is, that F^Y *is unknown* $\Longrightarrow$ *later.*

Definition

Let $P = \{\mathbb{P}_\theta \mid \theta \in \Theta\}$ be a statistical model with the corresponding joint distributions $\{F_\theta^{\mathsf{X},\mathsf{Y},\underline{\mathsf{Y}},\overline{\mathsf{Y}}} \mid \theta \in \Theta\}$ and $\mathsf{X}, \underline{\mathsf{Y}}, \overline{\mathsf{Y}}$ random variables with the joint distribution $F^{\mathsf{X},\underline{\mathsf{Y}},\overline{\mathsf{Y}}}$. The ***Sharp Estimation Region (SER)*** *is defined as:*

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If the model is correctly specified, this region can also be written as:

$$SER(\underline{Y}, \overline{Y}) = \underset{\theta \in \Theta}{\operatorname{argmin}} \left(\inf_{Y s.t. C(X,Y,\underline{Y},\overline{Y})=1} L\left(F_\theta, F^{X,Y,\overline{Y},\underline{Y}}\right)\right).$$

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Now: Linear Model

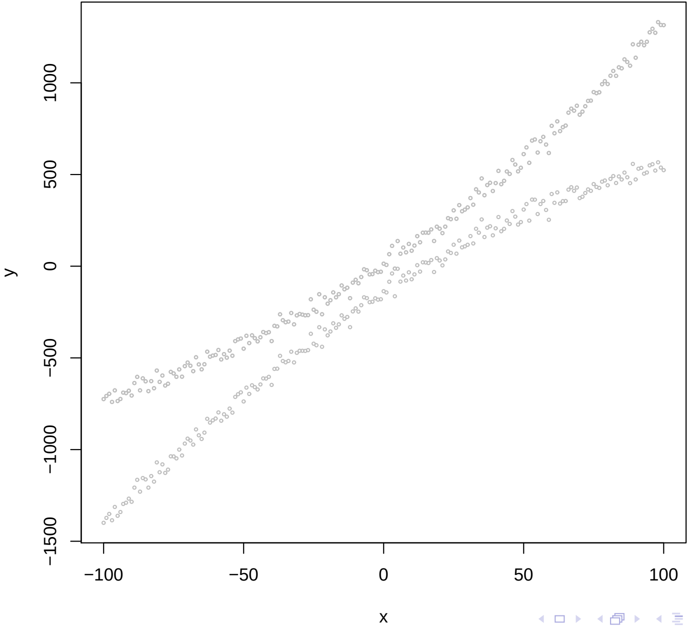
We are only interested in the components (β_0, β_1) of an element $\theta = ((\beta_0, \beta_1), \sigma^2, \sigma_l, \sigma_u) \in SER$ and denote the set

$$\{(\beta_0, \beta_1) \mid ((\beta_0, \beta_1), \sigma^2, \sigma_l, \sigma_u) \in SER\}$$

as the sharp estimation region (analogously for the sharp prediction region).

Linear Model

$$
\begin{aligned}
SER &= \{\beta \in B | \mathbb{E}(\underline{Y} \mid X) \le X\beta \le \mathbb{E}(\overline{Y} \mid X)\} \\
SPR &= \{\operatorname*{argmin}_{\beta \in B} \mathbb{E}((X\beta - Y)^2) \mid Y \in [\underline{Y}, \overline{Y}]\} \\
&= \{(X'X)^{-1}X'Y \mid Y \in [\underline{Y}, \overline{Y}]\}
\end{aligned}
$$



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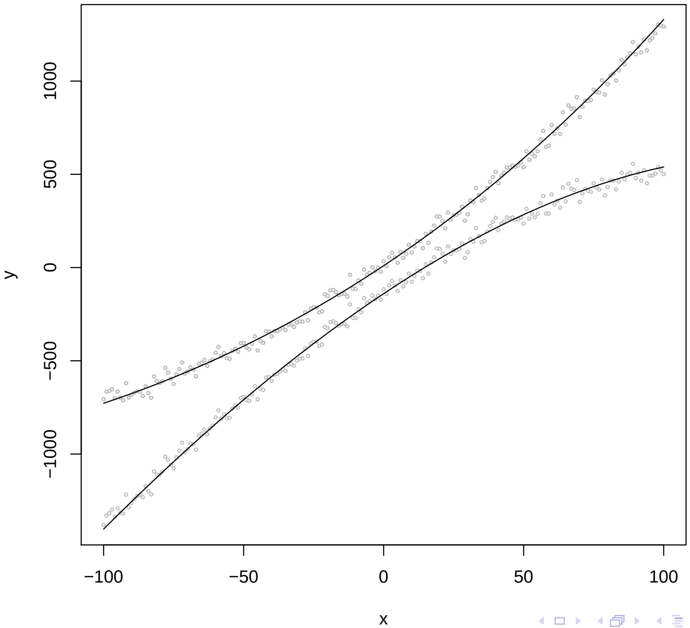

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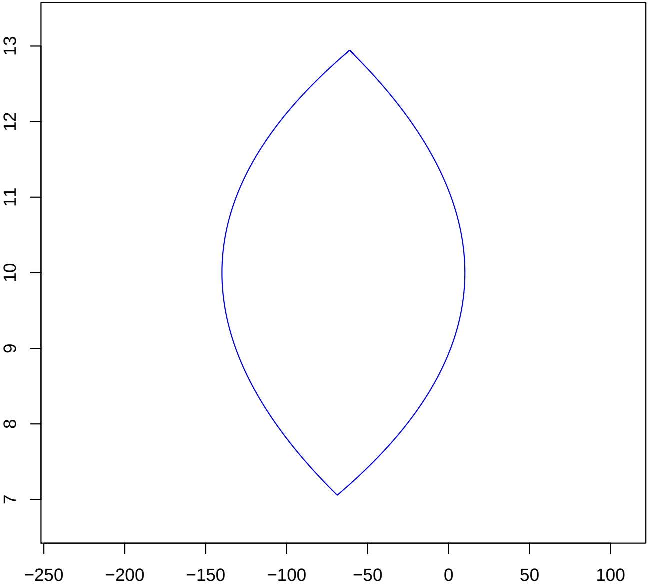

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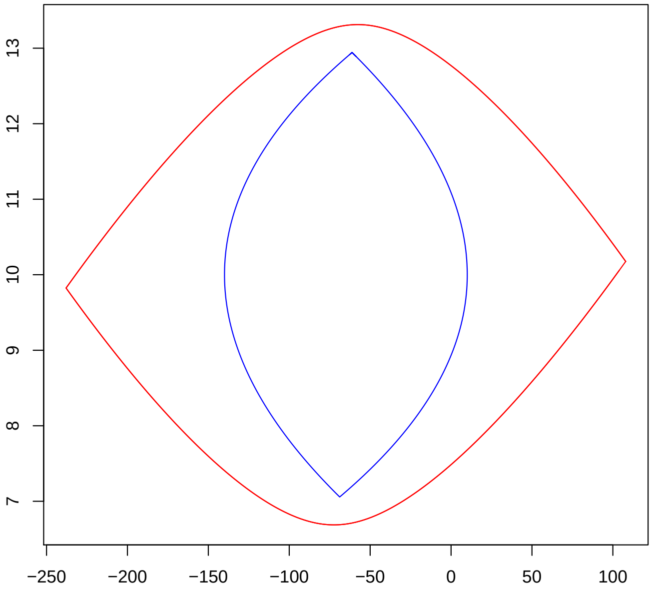

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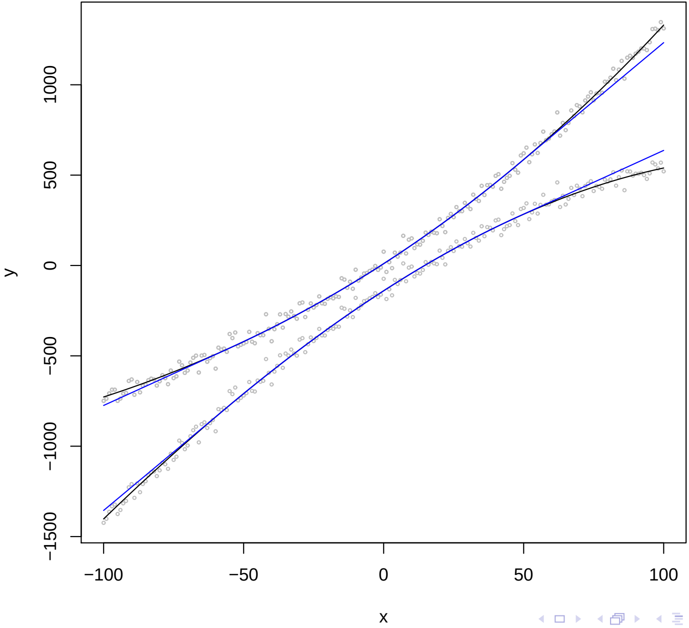
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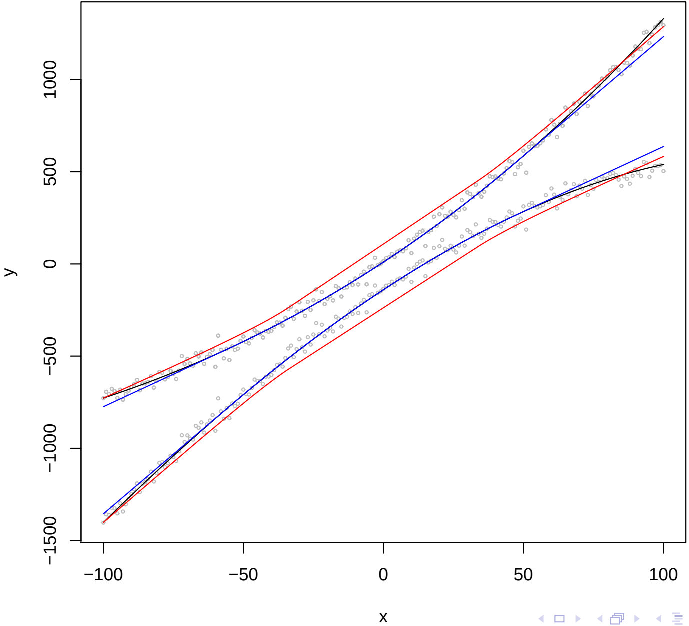
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Theorem

Let $I \subset \mathbb{R}^2$ be a compact convex set. Then there exist random variables $X, \underline{Y}, \overline{Y}$ such that

$$SER(X, \underline{Y}, \overline{Y}) = I,$$

namely:

$$\begin{aligned} X &\sim N(0,1) \\ \underline{Y} &= \min\{\beta_0 + \beta_1 X \mid (\beta_0, \beta_1) \in I\} \\ \underline{Y} &= \max\{\beta_0 + \beta_1 X \mid (\beta_0, \beta_1) \in I\}. \end{aligned}$$

Definition

The Minkowski-Sum

$$M = \bigoplus_{i=1}^{n} l_i = \left\{ \sum_{i=1}^{n} p_i \mid p_i \in l_i \right\}$$

*of n line-segments $l_i \subseteq \mathbb{R}^d$ is called a **zonotope**.*
A zonotope is a convex, compact and centrally symmetric polytope with finite many extremepoints and central-symmetric facets.

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Definition

*A closed, centrally symmetric convex set $Z \subseteq \mathbb{R}^d$ is called a **zonoid**, if it can be approximated arbitrarily closely by zonotopes (w.r.t. a metric, e.g. the Hausdorff distance).*
For $d = 2$ the zonoids are exactly the closed, centrally symmetric convex sets.

Lemma

Let $I \subseteq \mathbb{R}^2$ be a zonoid in general position. Then there exists random variables $X, \underline{Y}, \overline{Y}$ such that

$$SPR(X, \underline{Y}, \overline{Y}) = I.$$

Lemma

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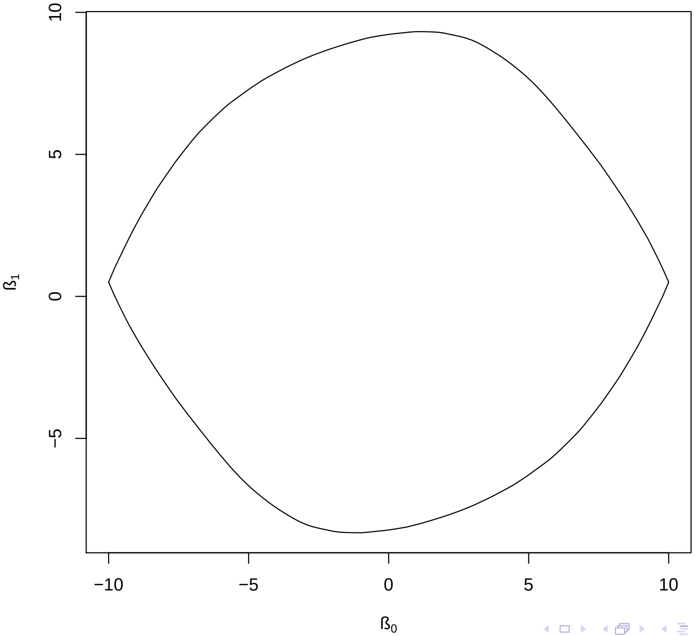
Lemma

Let $I = SPR(X, \underline{Y}^*, \overline{Y}^*) \subseteq \mathbb{R}^2$ be a zonoid and $E \subseteq SER(X, \underline{Y}^*, \overline{Y}^*)$ an arbitrary compact convex set. Then for every $\varepsilon > 0$ there exist random variables $X, \underline{Y}, \overline{Y}$ such that:

$$\begin{aligned} d_H(SPR(X, \underline{Y}, \overline{Y}), I) &\leq \varepsilon \\ d_H(SER(X, \underline{Y}, \overline{Y}), E) &\leq \varepsilon \end{aligned}$$

with the Hausdorff distance d_H.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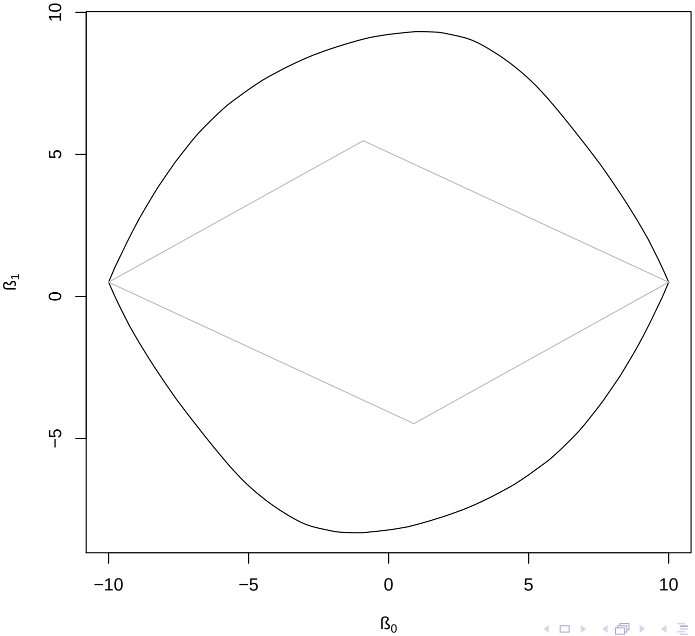
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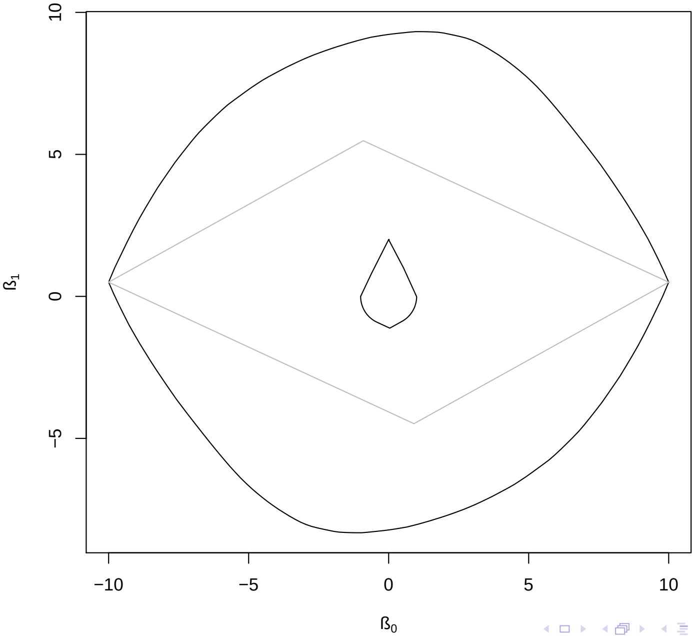
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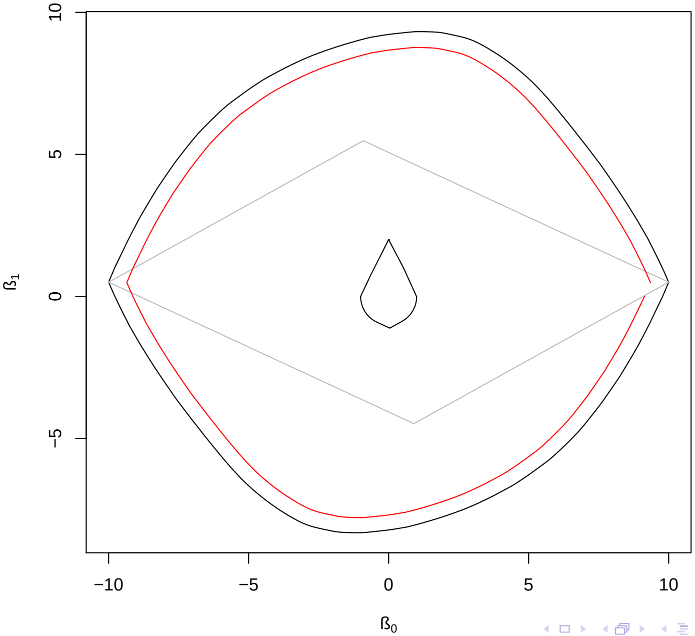

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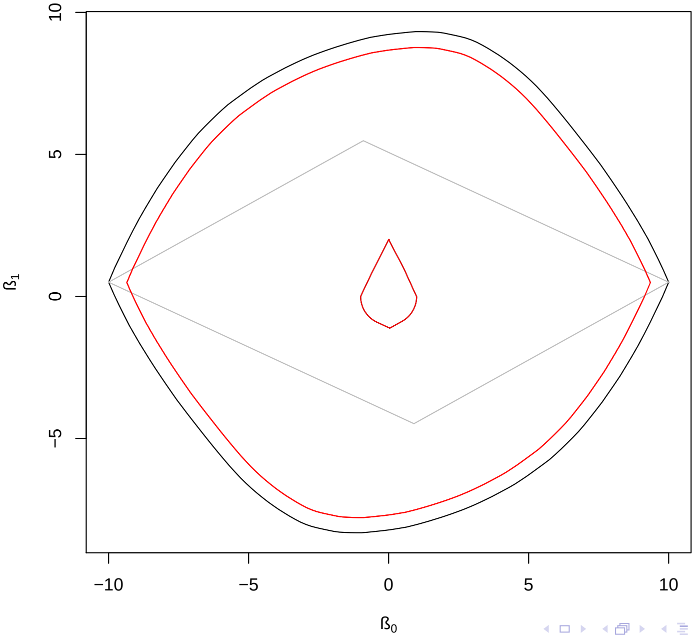
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Mappings between ordered sets

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Definition

Let $(P, \leq)$ and $(Q, \sqsubseteq)$ be partially ordered sets. A pair (f, g) of mappings $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ is called ***adjunction****, if:*

$$\forall p \in P \forall q \in Q : \quad p \leq g(q) \iff f(p) \sqsubseteq q.$$

In this case, f is called ***left adjoint*** *and g is called* ***right adjoint****.*

Examples of adjunctions

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- *Dempster-Shafer-Theory:*
  Multivalued mapping $\Gamma : X \longrightarrow 2^S$ *with corresponding*

  $$\tilde{\Gamma} : (2^X, \subseteq) \longrightarrow (2^S, \subseteq) : A \mapsto \bigcup_{a \in A} \Gamma(a) \textit{ and the operator}$$

  $$_* : (2^S, \subseteq) \longrightarrow (2^X, \subseteq) : T \mapsto \{x \in X \mid \Gamma(x) \subseteq T\}.$$

  The pair $(\tilde{\Gamma}, \, _*)$ *is an adjunction.*
  From this, the ∞*-monotonicity of a Belief-function*

  $$Bel = P \circ \, _*$$

  with P *a probability-measure follows immediately, since* P *is* ∞*-monotone and* $_*$ *is meet-preserving. Furthermure it is clear, that also* $Bel \circ \, _*$ *is* ∞*-monotone.*

Examples of adjunctions

- *Lower coherent previsions:*
  $f : \underline{P} \mapsto \mathcal{M}(\underline{P}) = \{p \in \mathscr{P}(\Omega) \mid p \geq \underline{P}\}$ *and*

  $g : M \mapsto \underline{P}_M : X \mapsto \inf_{p \in M} p(X)$ *are an adjunction.*

Examples of adjunctions

- *Formal concept analysis:*
  Incidence structure $\mathbb{K} = (G, M, I)$
  with $G \ldots$ *objects,* $M \ldots$ *attributes and a relation* $I \subseteq G \times M$*.*
  $(g, m) \in I$ *means object* g *has attribute* m *(also denotad as* gIm*).*

  $f : (2^M, \subseteq) \longrightarrow (2^G, \subseteq) : X \mapsto \{g \in G | \forall m \in X : gIm\}$

  „The set of all objects having all attributes in X*"*

  $g : (2^G, \supseteq) \longrightarrow (2^M, \supseteq) : Y \mapsto \{m \in M \mid \forall g \in Y : gIm\}$

  „The set of all joint attributes of all objects in Y*".*

  The pair (f, g) *is an adjunction.*

Lemma

Let (f, g) be an adjunction. Then the following holds:

A1 $g \circ f$ *is extensive and* $f \circ g$ *is intensive.*

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- A5 *The adjoints* f *and* g *are determining each other unambiguously.*
- A6 f *is join-preserving and* g *is meet-preserving.*

Lemma

- *If P is a complete lattice, than f is a left adjoint, if and only if f is join-preserving.*
- *If Q is a complete lattice, than g is a right adjoint, if and only if g is meet-preserving.*

Lemma

The mapping

$$SER \quad : \quad (\mathcal{Z}(\Omega), \leq) \longrightarrow (2^B, \subseteq) : (X, \underline{Y}, \overline{Y}) \mapsto \{\beta \mid \mathbb{E}(\underline{Y}\,|X) \leq \beta X \leq \mathbb{E}(\overline{Y}\,|x)\}$$

with

$$\boxed{(X_1, \underline{Y}_1, \overline{Y}_1) \leq (X_2, \underline{Y}_2, \overline{Y}_2):} \quad \Longleftrightarrow \quad \boxed{\mathbb{E}(\underline{Y}_1 \mid X) \geq \mathbb{E}(\underline{Y}_2 \mid X) \;\&\; \mathbb{E}(\overline{Y}_1 \mid X) \leq \mathbb{E}(\overline{Y}_2 \mid X)}$$

i.e.: $(X_1, \underline{Y}_1, \overline{Y}_1)$ *is more precise than* $(X_2, \underline{Y}_2, \overline{Y}_2)$

is a right adjoint.

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i.e.: $(X_1, \underline{Y}_1, \overline{Y}_1)$ *is more precise than* $(X_2, \underline{Y}_2, \overline{Y}_2)$

is a right adjoint.
The corresponding left adjoint is the „prediction-operator":

$$PR : \qquad (2^B, \subseteq) \longrightarrow \left(\mathcal{Z}(\Omega), \leq\right) : M \mapsto (X, \min_{\beta \in M} X\beta, \max_{\beta \in M} X\beta\right).$$

Lemma

Thus, the following holds:

A1 $SER \circ PR$ *is extensive and* $PR \circ SER$ *is intensive.*

A2 PR *and* SER *are order-preserving.*

A3 $PR \circ SER \circ PR = PR$ *and* $SER \circ PR \circ SER = SER$ *and thus* $PR \circ SER$ *and* $SER \circ PR$ *are idempotent.*

A4 *From A1 - A3 it follows, that* $SER \circ PR$ *is a hull operator and* $PR \circ SER$ *is a kernel operator.*

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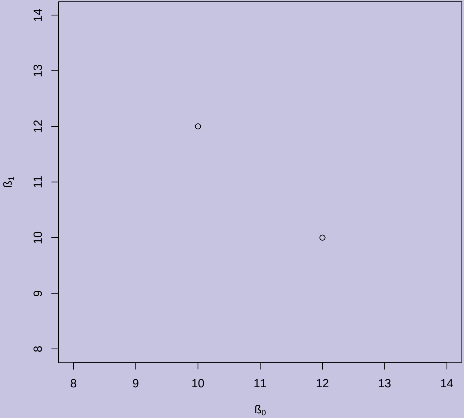
A6 PR *is join-preserving and* SER *is meet-preserving.*

Lemma

A1 *SER ∘ PR is extensive and PR ∘ SER is intensive.*



Lemma

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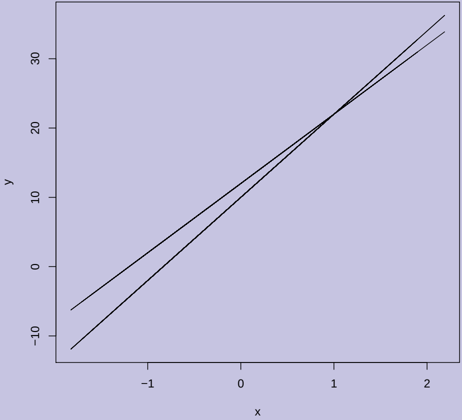

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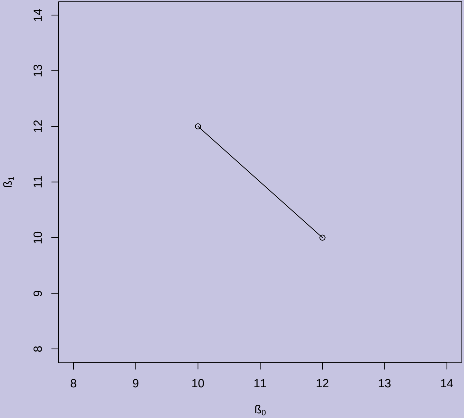

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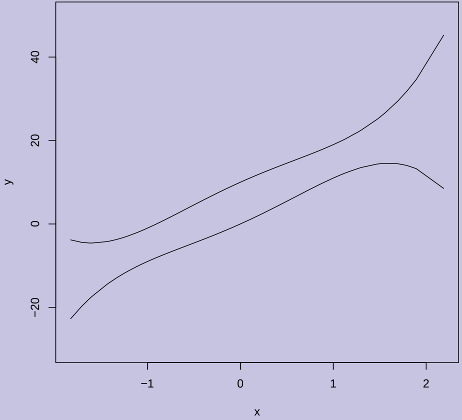

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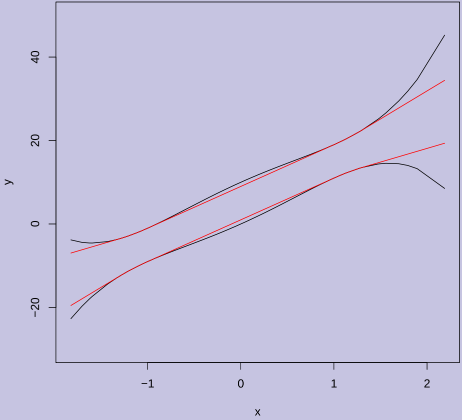


Lemma

Thus, the following holds:

A1 $SER \circ PR$ *is extensive and* $PR \circ SER$ *is intensive.*

A2 PR *and* SER *are order-preserving.*

A3 $PR \circ SER \circ PR = PR$ *and* $SER \circ PR \circ SER = SER$ *and thus* $PR \circ SER$ *and* $SER \circ PR$ *are idempotent.*

A4 *From A1 - A3 it follows, that* $SER \circ PR$ *is a hull operator and* $PR \circ SER$ *is a kernel operator.*

A5 *The adjoints* PR *and* SER *are determining each other unambiguously.*

A6 PR *is join-preserving and* SER *is meet-preserving.*

Lemma

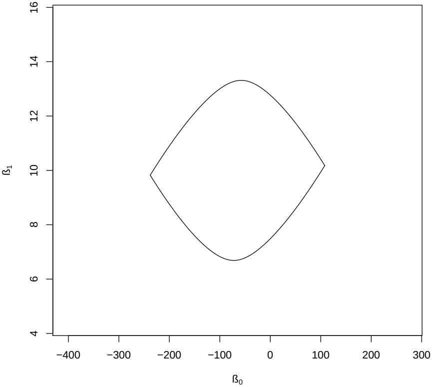
The mapping

$$SPR : (\mathcal{Z}(\Omega), \leq) \longrightarrow (2^B, \subseteq) : (X, \underline{Y}, \overline{Y}) \mapsto \{(X'X)^{-1}X'Y \mid \underline{Y} \leq Y \leq \overline{Y}\}$$

is no right adjoint, since it is not meet-preserving.

In general $SPR(Z_1 \wedge Z_2) \neq SPR(Z_1) \cap SPR(Z_2)$, *since the intersection of two zonoids is in general not a zonoid.*

Thus, in general, only $SPR \circ PR \circ SPR \supset SPR$ *holds.*



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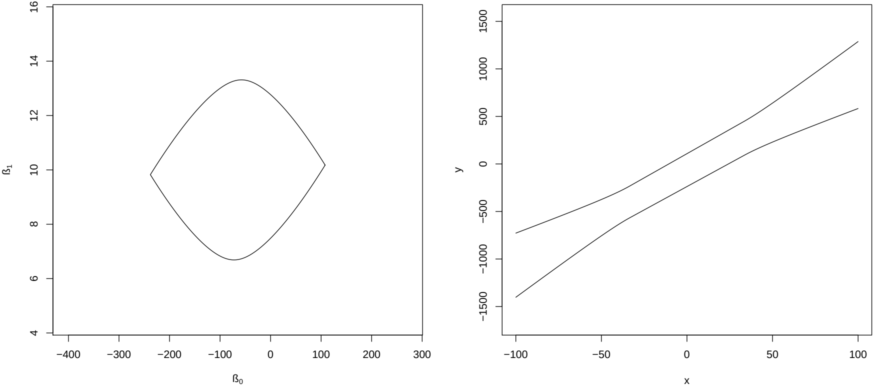

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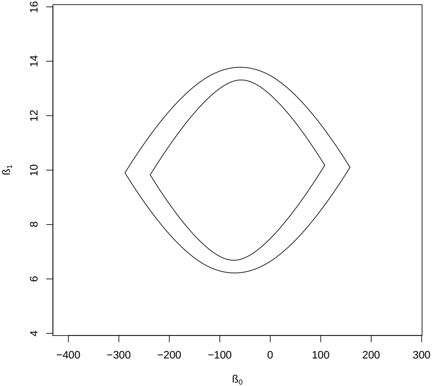

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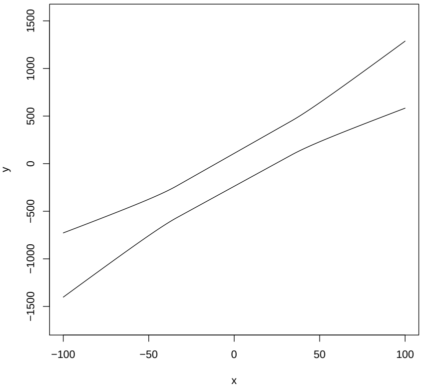

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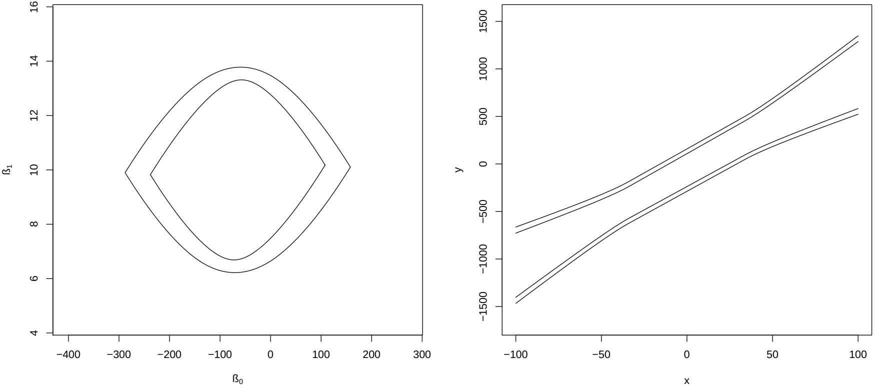

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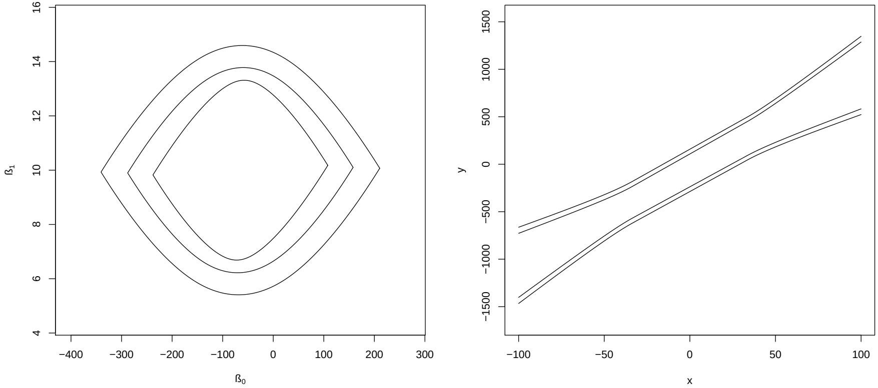

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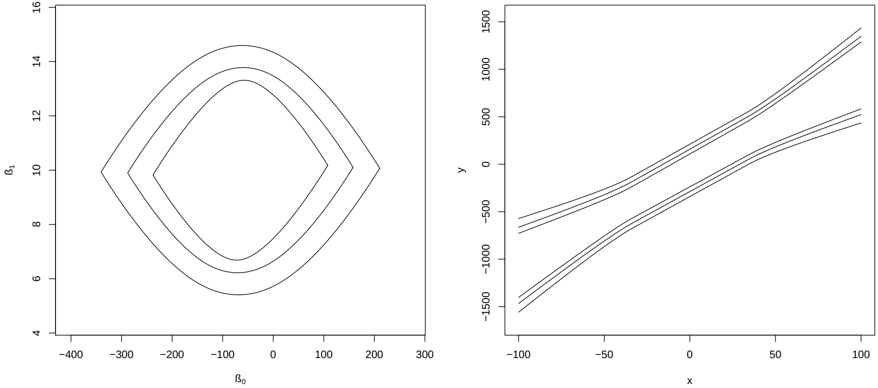

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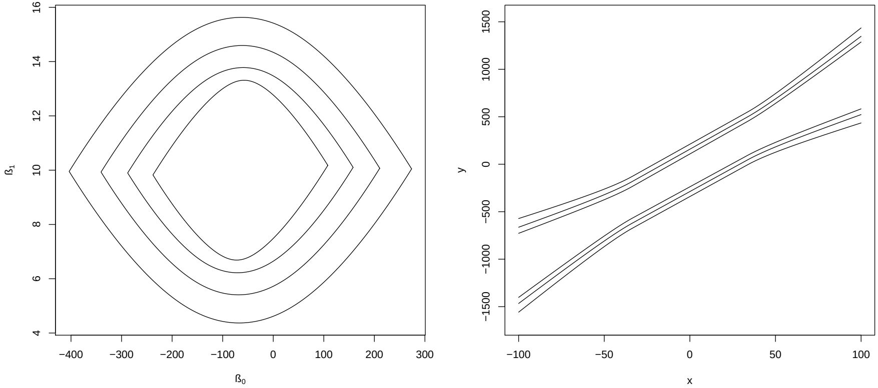

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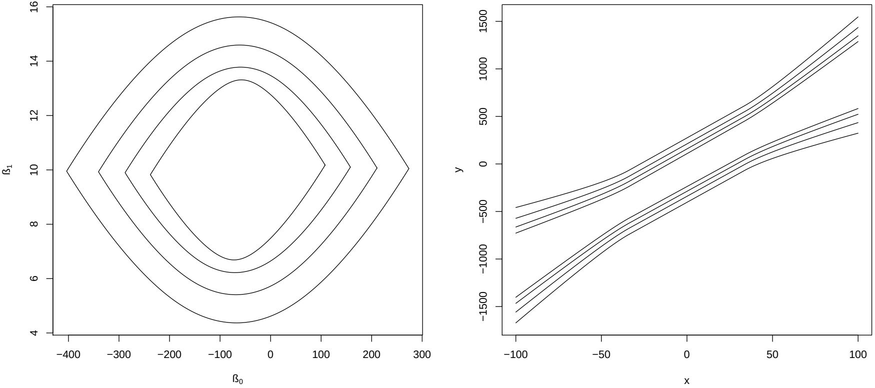
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Definition

Let $E : (P, \leq) \longrightarrow (Q, \sqsubseteq)$ *be a mapping.*
The monotone hull of E *is defined as:*

$$H(E) \quad : \quad (P, \leq) \longrightarrow (Q, \sqsubseteq) : X \mapsto \bigvee_{Y \leq X} E(Y).$$

The monotone kernel of E *is defined as:*

$$K(E) \quad : \quad (P, \leq) \longrightarrow (Q, \sqsubseteq) : X \mapsto \bigwedge_{Y \geq X} E(Y).$$

These set-valued mappings are both order-preserving
(i.e: $X \leq Y \Longrightarrow (H(E))(X) \sqsubseteq (H(E))(Y) \quad \& \quad (K(E))(X) \sqsubseteq (K(E))(Y)$*).*

A criterion-function-based mapping

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Lemma

Let the criterion-function $Q : B \longrightarrow \mathbb{R}$ *be defined as:*

$$Q(\beta) = \int \left\{ (\mathbb{E}(\underline{Y}\,|x) - x\beta)_+^2 + \left(\mathbb{E}(\overline{Y}\,|x) - x\beta\right)_-^2 \right\} d\mathbb{P}(x).$$

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Then the criterion-based mapping

$$E_Q : \mathcal{Z}(\Omega) \longrightarrow 2^B : (X, \underline{Y}, \overline{Y}) \mapsto \operatorname*{argmin}_{\beta \in B} Q(\beta)$$

is a source of SER and SPR:

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is a source of SER and SPR:

$$\begin{aligned} SPR &= H(E_Q) \\ SER &= K(E_Q). \end{aligned}$$

Estimation of SER and SPR

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Lemma

In general, there is no monotone, nonpartial, consistent estimator of SER.

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In general, there is no consistent and (in a certain sense) robust estimator of SER.

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