Robust regression with imprecise data

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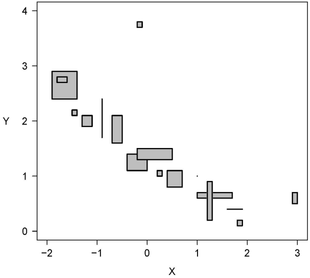
imprecise data

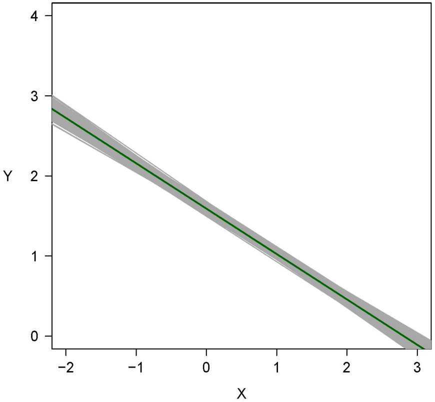
example data set:



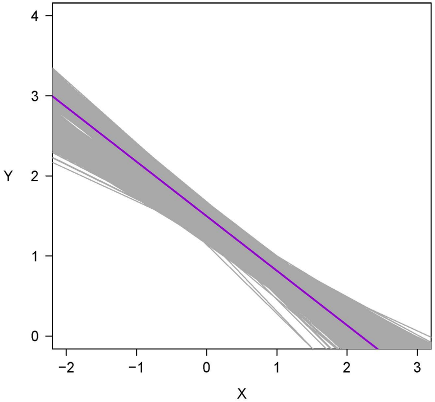
- in the literature, two kinds of general approaches to regression with imprecise data:
  - midpoint regression: represent the observed imprecise data by *few precise values* (e.g., intervals by midpoint and length), and apply standard regression methods to those values (see, e.g., Domingues et al., 2010)
  - set of precise regressions: apply standard regression methods to *all possible precise data* compatible with the observed imprecise data, and consider the range of outcomes as the imprecise result (see, e.g., Ferson et al., 2007)
- LIR (Likelihood-based Imprecise Regression): new regression method directly applicable to imprecise data (Cattaneo and Wiencierz, 2011*a*,*b*)

linear regression for the example data set

midpoint regression (green and violet lines) and set of precise regressions (set of gray lines):



least squares



least median of squares

nonparametric likelihood

- precise data (unobserved): random variables $V_i = (X_i, Y_i) \in \mathcal{X} \times \mathbb{R}$
- imprecise data (observed): random sets $V_i^* \subseteq \mathcal{X} \times \mathbb{R}$
- nonparametric model: $\mathcal{P}$ is the set of all probability measures such that
  - $(V_1, V_1^*), \ldots, (V_n, V_n^*)$ i.i.d.
  - $P(V_i \in V_i^*) \geq 1 - \varepsilon$ (where $\varepsilon \in [0, 1]$ is fixed)
- the observed (imprecise) data $V_1^* = A_1, \ldots, V_n^* = A_n$ induce the (normalized) likelihood function $lik : \mathcal{P} \to [0, 1]$ with

$$lik(P) = \frac{P(V_1^* = A_1, \ldots, V_n^* = A_n)}{\max_{P' \in \mathcal{P}} P'(V_1^* = A_1, \ldots, V_n^* = A_n)}$$

regression problem

- regression functions: $\mathcal{F}$ is a certain set of functions $f : \mathcal{X} \to \mathbb{R}$
- absolute residuals: $R_{f,i} = |Y_i - f(X_i)|$
- for each function $f \in \mathcal{F}$, the quantiles of the distribution of the absolute residuals $R_{f,i}$ can be estimated even under the nonparametric model $\mathcal{P}$
- the regression problem can be interpreted as the *minimization of the p-quantile* of the distribution of the absolute residuals $R_{f,i}$ (where $p \in (0, 1)$ is fixed)

generalized LQS regression

- likelihood-based confidence interval for the p-quantile of the distribution of the absolute residuals $R_{f,i}$ (where $Q_f(P)$ is the interval of all p-quantiles of $R_{f,i}$ under P, and $\beta \in (0,1)$ is fixed):

$$\mathcal{C}_f = \bigcup_{P \in \mathcal{P} \,:\, lik(P) > \beta} Q_f(P)$$

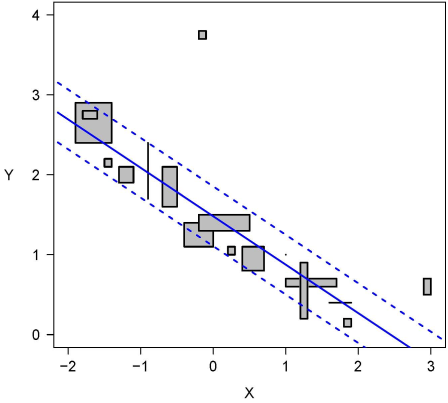
- *point estimate*: f_{LRM} is the function in $\mathcal{F}$ minimizing $\sup \mathcal{C}_f$ (Likelihood-based Region Minimax: see Cattaneo, 2007)
- f_{LRM} has a simple geometrical interpretation: $\overline{B}_{f_{LRM},\overline{q}_{LRM}}$ is the thinnest band of the form $\overline{B}_{f,q} = \{(x,y) \in \mathcal{X} \times \mathbb{R} : |y - f(x)| \leq q\}$ *containing* at least $\overline{k}$ imprecise data (where $\overline{k} > (p + \varepsilon)\, n$ depends on n, ε, p, β), for all $f \in \mathcal{F}$ and all $q \in [0, +\infty)$
- when the observed data are in fact precise, f_{LRM} corresponds to the LQS (Least Quantile of Squares) estimate with quantile $\frac{\overline{k}}{n}$
- f_{LRM} can be computed by generalizing the algorithm of Rousseeuw and Leroy (1987)

imprecise regression

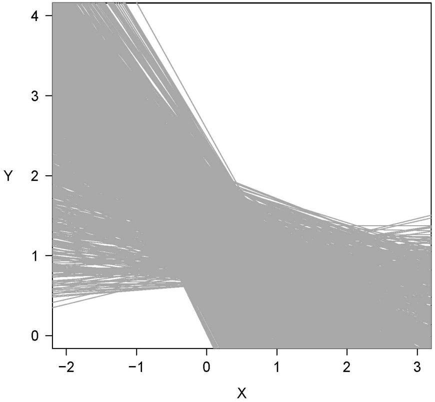
- interval dominance: interval I (strictly) dominates interval J iff $x < y$ for all $x \in I$ and all $y \in J$
- imprecise regression: set of *all undominated functions* (i.e., all $f \in \mathcal{F}$ such that $\overline{q}_{LRM} \in \mathcal{C}_f$)
- the undominated functions have a simple geometrical interpretation: f is undominated iff $\overline{B}_{f,\overline{q}_{LRM}}$ *intersects* at least $\underline{k}+1$ imprecise data (where $\underline{k} < (p-\varepsilon)\,n$ depends on n, ε, p, β)
- *complex uncertainty*, consisting of two kinds of uncertainty:
  - statistical uncertainty: decreases when n increases (reflected by the spread between $\frac{\underline{k}+1}{n}$ and $\frac{\overline{k}}{n}$)
  - indetermination: unavoidable under such weak assumptions (reflected by the difference between *containing* and *intersecting* imprecise data)

LIR analysis of the example data set

$$n = 17, \quad \varepsilon = 0, \quad p = 0.5, \quad \beta = 0.5 \quad \Rightarrow \quad \overline{k} = 11, \quad \underline{k} = 6$$



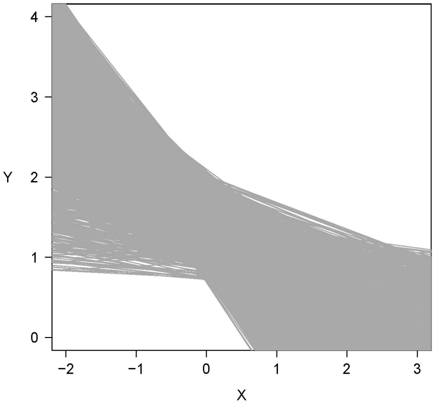
f_{LRM} and $\overline{B}_{f_{LRM},\overline{q}_{LRM}}$



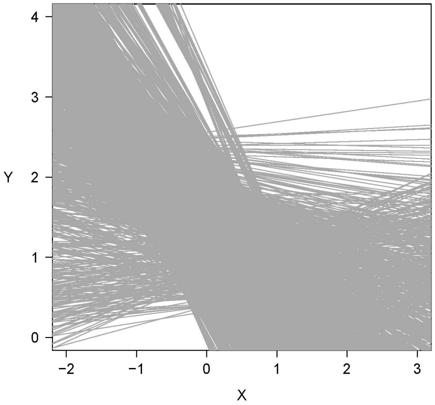
undominated lines ($\overline{k} = 11$, $\underline{k} = 6$)

other values for β and ϵ

$$n = 17, \quad \varepsilon = 0, \quad p = 0.5, \quad \beta = 0.8 \quad \Rightarrow \quad \overline{k} = 10, \quad \underline{k} = 7$$
$$n = 17, \quad \varepsilon = 0.05, \quad p = 0.5, \quad \beta = 0.8 \quad \Rightarrow \quad \overline{k} = 11, \quad \underline{k} = 6$$
$$n = 17, \quad \varepsilon = 0, \quad p = 0.5, \quad \beta = 0.26 \quad \Rightarrow \quad \overline{k} = 12, \quad \underline{k} = 5$$
$$n = 17, \quad \varepsilon = 0.05, \quad p = 0.5, \quad \beta = 0.5 \quad \Rightarrow \quad \overline{k} = 12, \quad \underline{k} = 5$$



undominated lines ($\overline{k} = 10,\ \underline{k} = 7$)

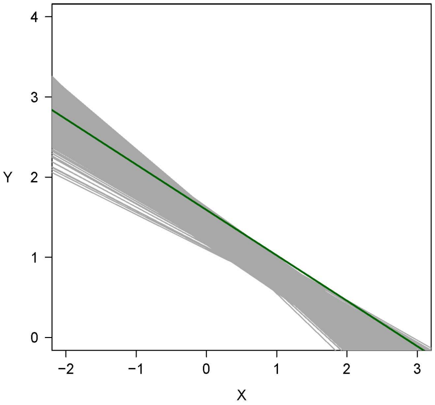


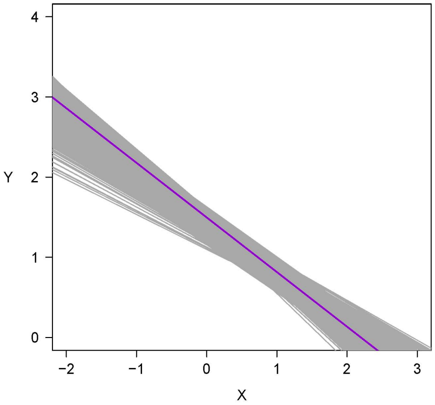
undominated lines ($\overline{k} = 12,\ \underline{k} = 5$)

precise data (midpoints of the example data set)

$n = 17, \quad \varepsilon = 0, \quad p = 0.5, \quad \beta = 0.5 \quad \Rightarrow \quad \overline{k} = 11, \quad \underline{k} = 6$

LIR analysis (undominated functions: gray lines), least squares regression (green line), and least median of squares regression (violet line):



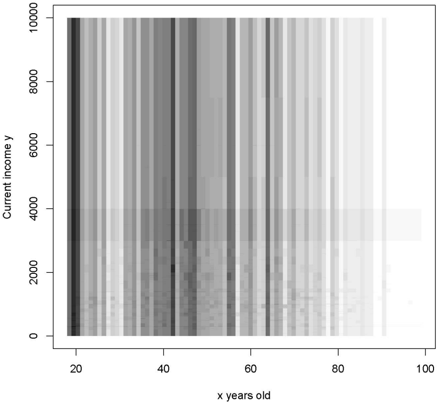


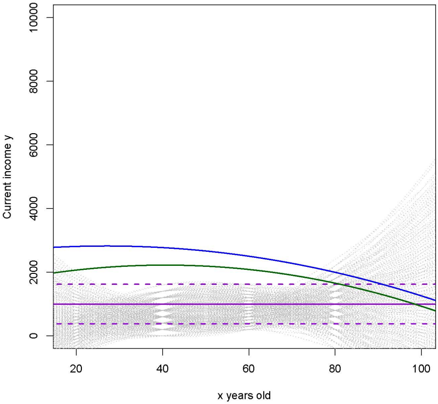
example with social survey data

- data from the "ALLBUS — German General Social Survey" of 2008 (provided by GESIS — Leibniz Institute for the Social Sciences)
- relationship between age $X_i \in \mathcal{X} = [18, 100)$ and personal income (on average per month) $Y_i \in [0, +\infty)$, with $n = 3469$
- choice of regression functions: $\mathcal{F} = \{f_{a,b_1,b_2} : a, b_1, b_2 \in \mathbb{R}\}$ is the set of all quadratic functions $f_{a,b_1,b_2}(x) = a + b_1\,x + b_2\,x^2$
- choice of parameters:
  - $\varepsilon = 0, \quad p = 0.5, \quad \beta = 0.15 \quad \Rightarrow \quad \overline{k} = 1792, \quad \underline{k} = 1677$
  - $\varepsilon = 0.0107, \quad p = 0.5, \quad \beta = 0.8 \quad \Rightarrow \quad \overline{k} = 1792, \quad \underline{k} = 1677$
- in 4 different data situations, f_{LRM} (violet solid line, with $\overline{B}_{f_{LRM},\overline{q}_{LRM}}$ represented by the violet dashed lines) and the undominated functions (gray dotted curves) are compared with the results of the least squares midpoint regressions with upper income limit 15000 (blue curve) or 10000 (green curve)

original data

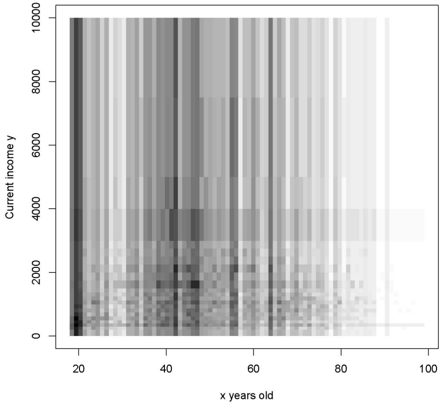
- age data: 3457 “precise” (in years: 83 classes), 12 missing
- income data: 2406 precise, 381 categorized (22 classes), 682 missing

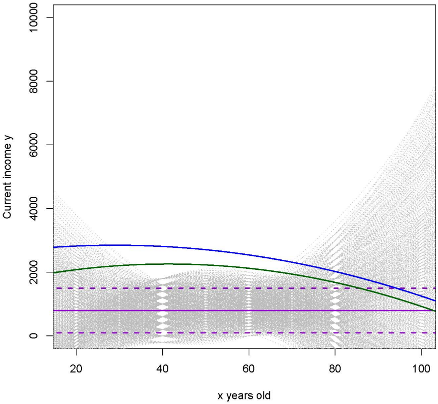




categorized income data

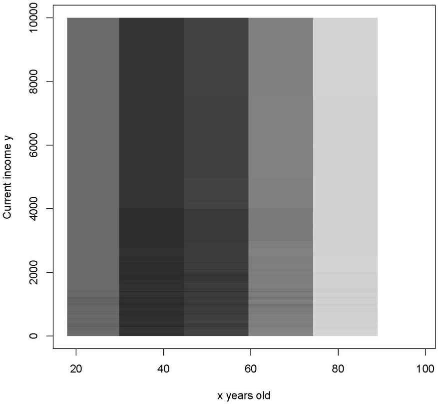
- age data: 3457 "precise" (in years: 83 classes), 12 missing
- income data: 2787 categorized (22 classes), 682 missing

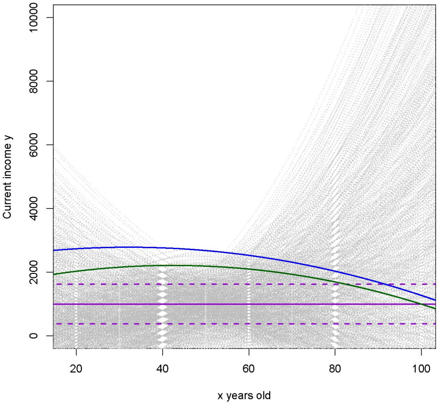




categorized age data

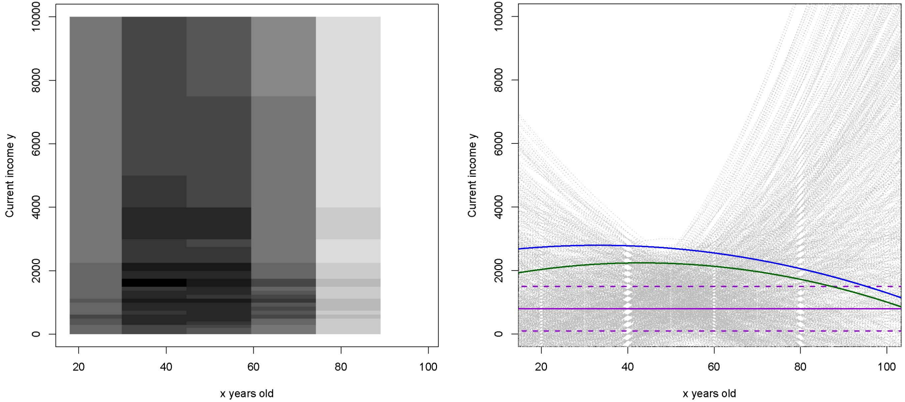
- age data: 3457 categorized (6 classes), 12 missing
- income data: 2406 precise, 381 categorized (22 classes), 682 missing





categorized age and income data

- age data: 3457 categorized (6 classes), 12 missing
- income data: 2787 categorized (22 classes), 682 missing



conclusion and outlook

- LIR: new line of approach to regression with imprecise data
- LIR is directly applicable to any kind of imprecise data (with precise data as a special case), where the *coarsening process* can be informative (and even wrong with a certain probability)
- the result of the regression is imprecise, reflecting the total uncertainty in the data
- LIR without distributional assumptions leads to very robust results (generalized LQS regression)
- future work:
  - improve the *implementation* of LIR
  - study in more detail the *statistical properties* of the method (e.g., the coverage probability of the imprecise result), even though the repeated sampling evaluation is particularly problematic with imprecise data
  - investigate the consequences of *stronger assumptions* (e.g., the existence of a true regression function with an additive, homoscedastic, normal error)
  - consider the minimization of other properties of the distribution of the absolute residuals (besides the quantiles), in order to increase the *efficiency* of the method (e.g., generalized LTS regression)

references

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