Maxitive Integral of Real-Valued Functions

Marco Cattaneo
Department of Statistics, LMU Munich

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    (Neyman and Pearson, 1928; Cattaneo, 2013)
- to avoid trivial results, we assume that $0 < \mu(C) < 1$ for some $C \subset \Omega$
  (in particular, μ cannot be additive and maxitive at the same time)

extensions of capacities

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- to simplify the results, we consider only extensions F that are:
  - monotonic: $f \leq g \;\Rightarrow\; F(f) \leq F(g)$
  - calibrated: $\alpha \in \mathbb{R} \;\Rightarrow\; F(\alpha\, I_\Omega) = \alpha$
  - null preserving: $\mu\{f \neq 0\} = 0 \;\Rightarrow\; F(f) = 0$

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  - scale invariant: $\beta \in \mathbb{R}_{>0} \;\Rightarrow\; F(\beta f) = \beta F(f)$
  - location invariant: $\alpha \in \mathbb{R} \;\Rightarrow\; F(f + \alpha) = F(f) + \alpha$
  - convex: $\lambda \in (0,1) \;\Rightarrow\; F(\lambda f + (1-\lambda) g) \leq \lambda F(f) + (1-\lambda) F(g)$

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    i.e., the loss or utility of possible decisions can be measured on an interval scale (Stevens, 1946), since the location of the zero point is of no concern in the decision making
  - convex: $\lambda \in (0,1) \;\Rightarrow\; F(\lambda f + (1-\lambda)\, g) \leq \lambda F(f) + (1-\lambda)\, F(g)$,
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- when Ω is finite and μ is additive, the integral with respect to μ is a **weighted average**:

$$\int f \,\mathrm{d}\mu = \sum_{\omega \in \Omega} f(\omega)\, \mu\{\omega\}$$

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- when μ is maxitive, its **unique** scale invariant, maxitive extension to $\mathcal{B}^+$ (the set of all bounded functions $f : \Omega \to \mathbb{R}_{\geq 0}$) is the Shilkret integral with respect to μ, which is also convex (Shilkret, 1971):

$$\int^{\mathrm{S}} f \, \mathrm{d}\mu = \bigvee_{x \in \mathbb{R}_{>0}} x \, \mu\{f > x\}$$

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- when Ω is finite and μ is maxitive, the Shilkret integral with respect to μ is a **weighted maximum**:

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- when μ is maxitive, its **unique** location invariant, maxitive extension to $\mathcal{B}$ is the following integral with respect to μ, which is also convex and is therefore called convex integral:

$$\int^{\mathsf{X}} f \, \mathrm{d}\mu = \bigvee_{x \in \mathbb{R} \,:\, \mu\{f > x\} > 0} (x + \mu\{f > x\} - 1)$$

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- when Ω is finite and μ is maxitive, the convex integral with respect to μ is a **penalized maximum**:

$$\int^{\mathsf{X}} f \, \mathrm{d}\mu = \bigvee_{\omega \in \Omega \,:\, \mu\{\omega\} > 0} (f(\omega) + \mu\{\omega\} - 1)$$

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