Density Ratio Class Models and Imprecision

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The Density Ratio Class a.k.a. Interval of Measures

Define a set of priors $\mathcal{M}$ by

$$\mathcal{M}_{l,u} = \Big\{ p(\vartheta) : \exists c > 0 : l(\vartheta) \leq cp(\vartheta) \leq u(\vartheta) \Big\},$$

where the *lower and upper density functions* $l(\vartheta)$ and $u(\vartheta)$
are bounded non-negative functions for which $l(\vartheta) \leq u(\vartheta)\ \forall\ \vartheta \in \Theta$.

If $l(\vartheta) > 0\ \forall\ \vartheta$, then

$$\mathcal{M}_{l,u} = \left\{ p(\cdot) : \frac{p(\vartheta)}{p(\vartheta')} \leq \frac{u(\vartheta)}{l(\vartheta')}\ \forall\ \vartheta, \vartheta' \right\},$$

hence the name ‘density ration class’ [4, 1].

Properties (see, e.g., [6, §4.2.2])

- $\mathcal{M}_{\lambda l,\lambda u} = \mathcal{M}_{l,u} \; \forall \lambda > 0$
- Invariance under updating: set of posteriors via GBR is again a density ratio class $\mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}$, with lower and upper density functions the posteriors based on $l(\vartheta)$ and $u(\vartheta)$.
- Update of $l(\vartheta)$ and $u(\vartheta)$ can be done by updating a single $p(\vartheta) \in \mathcal{M}_{l,u}$ and then reweighting it to get $l(\vartheta \mid \boldsymbol{x})$ and $u(\vartheta \mid \boldsymbol{x})$.
- Closed-form expressions for posterior inferences, e.g.:

$$\underline{\mathrm{P}}_{l,u}(A \mid \boldsymbol{x}) = \min_{p \in \mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}} \mathrm{P}_p(A) = \left[1 + \frac{\int_{A^c} u(\vartheta \mid \boldsymbol{x}) \, \mathrm{d}\vartheta}{\int_A l(\vartheta \mid \boldsymbol{x}) \, \mathrm{d}\vartheta} \right]^{-1}$$

$$\overline{\mathrm{P}}_{l,u}(A \mid \boldsymbol{x}) = \max_{p \in \mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}} \mathrm{P}_p(A) = \left[1 + \frac{\int_{A^c} l(\vartheta \mid \boldsymbol{x}) \, \mathrm{d}\vartheta}{\int_A u(\vartheta \mid \boldsymbol{x}) \, \mathrm{d}\vartheta} \right]^{-1}$$

Imprecision

- Posterior bounding functions $l(\vartheta \mid \boldsymbol{x})$ and $u(\vartheta \mid \boldsymbol{x})$ will be more pointed, but imprecision of $\mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}$ is the same as $\mathcal{M}_{l,u}$:

$$\frac{u(\vartheta \mid \boldsymbol{x})}{l(\vartheta \mid \boldsymbol{x})} = \frac{f(\boldsymbol{x} \mid \vartheta)u(\vartheta)}{f(\boldsymbol{x} \mid \vartheta)l(\vartheta)} = \frac{u(\vartheta)}{l(\vartheta)}$$

- $\mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}$ does not converge to a one-element set for $n \to \infty$: there is never enough data for prior imprecision to vanish!

Density Class Ratio Models

- Rinderknecht et al. [6]:
  - Expert elicitation of $\mathcal{M}_{l,u}$ (given parametric families for l and u) based on probability-quantile (-interval) pairs.
  - Approximations to $l(\vartheta \mid \boldsymbol{x})$ and $u(\vartheta \mid \boldsymbol{x})$ by MCMC.

➡ High posterior imprecision in applications examples.

- Pericchi & Walley [5]:
  - Class with $l(\vartheta) \propto \mathrm{N}(\mu, \sigma^2)$ and $u(\vartheta) \propto 1$, where $l(\vartheta) = u(\vartheta)$ at $\vartheta = \mu$.
  - All $p \in \mathcal{M}_{l,u}$ must thus have their mode at μ.

➡ Reasonable imprecision behavior in case of prior-data conflict.

Imprecision in Pericchi & Walley model

- Imprecision inceases in $|\bar{x} - \mu|$ for fixed n
  ➡ prior-data conflict sensitivity
- Imprecision decreases in n when $\bar{x} = \mu$
- Imprecision remains approximately constant when $\bar{x} \neq \mu$
  ➡ same behaviour as in Rinderknecht examples
- Imprecision decreases in $\bar{x} = \mu$ case because all $p \in \mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}$ concentrate their mass at μ, where $l(\vartheta \mid \boldsymbol{x}) \approx u(\vartheta \mid \boldsymbol{x})$.
  ➡ you need $l(\vartheta) \approx u(\vartheta)$ for some ϑ for decreasing imprecision
- Other ways to have decreasing imprecision?

Models by Coolen [2, 3]

Let $u(\vartheta) = c_0 \cdot l(\vartheta)$, where $c_0 > 1$ constant, and

$l(\vartheta) = l(\vartheta \mid \psi^{(0)})$ be the conjugate prior with hyperparameter $\psi^{(0)}$.

Then $l(\vartheta \mid \boldsymbol{x}, \psi^{(0)}) = l(\vartheta \mid \psi^{(0)}) f(\boldsymbol{x} \mid \vartheta) = l(\vartheta \mid \psi^{(n)})$, and define

$$u(\vartheta \mid \boldsymbol{x}, \psi^{(0)}) =: \frac{c_n}{c_0} u(\vartheta \mid \psi^{(0)}) f(\boldsymbol{x} \mid \vartheta) = c_n l(\vartheta \mid \psi^{(n)}),$$

where c_n is introduced to let imprecision of $\mathcal{M}_{l,u}$ decrease with n.

Proposal of Coolen [2] for c_n such that $c_n \to 1$ for $n \to \infty$.

- No prior-data conflict sensitivity, because c_0 may not depend on ϑ.
- When instead different shapes are allowed for $l(\vartheta)$ and $u(\vartheta)$ [3], similar behaviour as previous models.
- Update $\mathcal{M}_{l,u} \longrightarrow \mathcal{M}_{l|\boldsymbol{x},u|\boldsymbol{x}}$ violates the GBR!

Suggestion

Combine ideas from Pericchi & Walley, Coolen, and Rinderknecht?

- Have $l(\vartheta) \approx u(\vartheta)$ for some ϑ.
- Reduce posterior imprecision by having a $c_n \to 1$ for $n \to \infty$.
- Elicit (and process?) $\mathcal{M}_{l,u}$ similar to Rinderknecht.

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